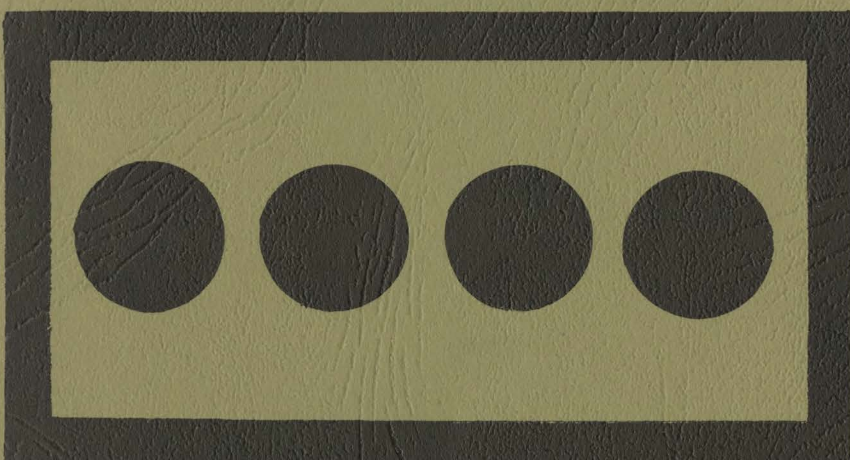
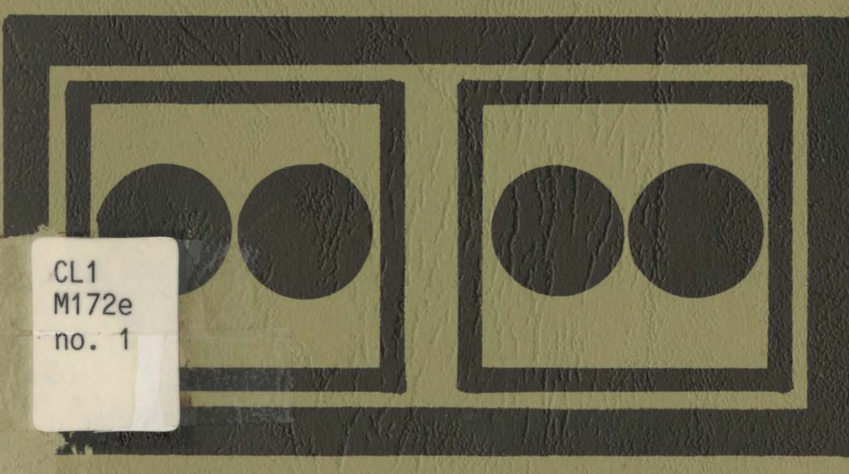
GUIDE NUMBER ONE

elementary mathematics

sets · system of natural and whole numbers · problem solving

E. T. McSwain





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for

The Piedmont Association for School Studies and Services
The University of North Carolina at Greensboro
Greensboro, North Carolina 27412

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SETS ▪ SYSTEM OF NATURAL AND WHOLE NUMBERS ▪ MATHEMATICAL OPERATIONS ▪ MATHEMATICAL PROPERTIES ▪ PROBLEM SOLVING ▪ MEASURES

I. **Introduction:** The purpose of this guide is to assist teachers in formulating consensus concerning important outcomes in teaching elementary mathematics. An understanding of elementary mathematics, frequently called modern mathematics, is an essential foundation for productive progress in mathematics in the junior high school and senior high school years.

II. **Desired Outcomes:** Some of the important outcomes in teaching elementary mathematics are:

A. To offer instruction and resourceful experiences that will encourage pupils to explore, to question, and to experiment as cognitive processes for discovering and understanding mathematical concepts and operations.

B. To help pupils to develop an understanding of the language of mathematics and its function in finding a solution for various problems encountered in contemporary society.

C. To assist pupils in learning the value of precision in interpreting and in applying mathematical ideas and operations.

D. To help pupils to understand that mathematics is an abstract language. The concepts, operations, and properties have meaning and function only in the mind of each pupil.

E. To help pupils to understand that an operation on a pair of numbers, such as addition and multiplication, is a mental operation governed by established mathematical properties or principles.

F. To encourage pupils to discover that self-motivation and reasonable time are important in learning mathematical concepts, operations, and properties.

III. **Mathematical Concepts Related to Sets:**

A. Numbers are abstract concepts derived from working with and thinking about sets of things.

B. A set is a collection of objects or symbols that has a description or classification, such as:

1. A set of books
2. A set of numerals
3. A set of pictures

C. The things contained in a set are called elements or members.

1. It is important that pupils understand the difference between a set and the elements contained in the set.

D. A set may be determined by naming its elements or members, or by describing them by some property they have in common.

E. It is important that pupils understand that mathematical concepts are not contained in a set. The concepts are mental ideas.

F. Some of the important mathematical concepts (ideas) that pupils should learn to apply to a set or sets are:

1. One-to-one matching of elements in two or more sets is a way to derive important mathematical ideas.
2. A cardinal number is the cardinality property or count assigned to a set.
3. Two sets containing the same number of like or common elements are called identical sets.
4. Two sets that have the same number of elements are called equivalent sets.
5. A set that contains no elements is called an empty set.
6. One-to-one mapping or counting determines when two sets of elements are equal, or greater than, or less than.

7. There is a one-to-one correspondence between standard sets and the ordered sequence of natural and whole numbers.

$\boxed{} = 0$	$\boxed{\cdot} = 1$	$\boxed{\cdot\cdot} = 2$	$\boxed{\vdots\,\cdot} = 3$	$\boxed{::} = 4$	$\boxed{::::::::::} = 10$	...
0	1	2	3	4	10	
zero	one	two	three	four	ten	

8. A number is a mathematical idea. A numeral is a name for a number.
9. There is an ordinal placement assigned to elements in a set. The placement names are: 1st or first; 2nd or second; 3rd or third; . . .
10. There is a basic difference between operations on sets, (such as, union of two sets, and intersection of two sets) and operations on numbers assigned to sets. The two operations on a pair of numbers are addition and multiplication. (Subtraction is the inverse operation of addition. Division is the inverse operation of multiplication.)
11. Elements in a set are enclosed by a pair of brackets { }, or by a VENN rectangle $\boxed{\quad\quad}$.
12. Capital letters are used to name a set.
    A = {1, 2, 3}   D = {John, Mary, Bill}

IV. Important Mathematical Concepts Related to Natural and Whole Numbers

A. The natural numbers are called the counting numbers.

1. Each number and its numeral represents the count or cardinal property assigned to a standard set.
2. The system of natural numbers begin with one. There is no zero.
3. The system of whole numbers includes zero and there is no last whole number. The system is infinite.
   Whole numbers = {0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, . . .}
4. There is an increase by one ordered sequence for the natural and whole numbers.

B. It is important that pupils understand that numeration is a way to write numerals that are the names for numbers.

1. The decimal (Hindu-Arabic) numeration system has ten as its base and has ten standard digits: 0,1,2,3,4,5,6,7,8,9.
2. The place-value positions are in an ordered sequence:
   a. The first position has the value, **one**.
   b. The next position, to the left of the one's place, has the value of ten.
   c. Each place-value position, to the left of the ten's place-value position, is an ordered power of ten.

THOUSAND	HUNDRED	TEN	ONE
$10 \times 10 \times 10$	10×10	10	1
1,000	100	10	1

3. Each digit of a numeral is assigned two values:
   a. The face-value or the count of the units.
   b. Place-value or the value of the position containing the digit.
   c. The digits of a numeral are read from left to right.
      (1) 345 is read 3 hundreds, 4 tens, 5 ones. $345 = 3(100) + 4(10) + 5(1)$
      (2) The greatest number of units that can be expressed in any place-value position is 9.
      (3) The absence of a unit in any place-value position is expressed by 0.
      (4) $708 = 7(100) + 0(10) + 8$
      (5) Standard notation for a whole number is the simplest way to name a number. 763 = 7 hundreds, 6 tens, 3 ones
      (6) Expanded notation requires the use of a numeral to express the decimal value of each place-value position.

$$763 = 7(100) + 6(10) + 3(1)$$

(7) Exponential notation expresses place-value position by using 10 and an exponent to name the power of ten.

$763 = 7(10)^2 + 6(10)^1 + 3(10)^0 \quad \{(10)^0 = 1\}$

C. Teachers of the primary grades should encourage pupils:

2. To prepare models to show the relation between sets, numbers, and numerals.

5   14
five   fourteen

2. To prepare a numeral chart for numbers from zero to one hundred as an aid in understanding their numerical relation.

V. Mathematical Concepts and Properties Related to Operations on Whole Numbers. Addition and Subtraction.

A. Addition.

1. Addition is a mathematical operation on a pair of whole numbers to find the equivalent standard number.
   a. $8 + 7 = \square$ b. $8 + 7 = 15$
   c. 15 is the standard or simplest name for $8 + 7$.
   d. $8 + 7$ is an expression that names the number fifteen.
   e. The symbol "$=$" is intepreted as **is** or **equals.**
2. It is imperative that pupils understand that addition is a mathematical operation on a pair of whole numbers.
   a. Pupils, when adding three or more numbers should understand that they add by pairs of numbers.
   $7 + 8 + 9 = \square \qquad (7 + 8) + 9 = \square \qquad 15 + 9 = 24$
3. Pupils, when using the decimal grid algorism, should understand that they add by pairs the numbers named in each place-value position.

Th	H	T	O
	3	8	6
	4	5	3
+	9	7	5
1	8	1	4

a. $6 + 3 = 9$
b. $9 + 5 = 1$ ten and 4 ones
c. 1 ten $+$ 8 tens $=$ 9 tens
d. $9T + 5T = 14T = 10T + 4T = 1H + 4T$
e. $4T + 7T = 11T = 10T + 1T = 1H + 1T$
f. $2H + 3H = 5H$
g. $5H + 4H = 9H$
h. $9H + 9H = 18H = 10H + 8H = 1Th + 8H$
$1{,}814 = 1Th + 8H + 1T + 4$ ones

4. Pupils who have not developed a mastery of the 100 standard combinations for addition on whole numbers will resort, frequently, to counting.
5. Pupils in the primary grades should be encouraged to write the pairs of one-digit numbers for each number from 1 to 18, i.e. . . .
   a. $1 = 1 + 0, 0 + 1$
   b. $10 = 9 + 1, 1 + 9, 8 + 2, 2 + 8,$
   $7 + 3, 3 + 7, 6 + 4, 4 + 6, 5 + 5$
   c. $18 = 9 + 9$
6. Teachers are expected to help pupils to discover and to understand the mathematical properties for addition.
   a. Identity property $0 + 1 = 1 \quad 1 + 0 = 1$
   b. Closure property $8 + 7 = 15$
   c. Commutative property $8 + 7 = 7 + 8$
   d. Associative property $9 + 7 + 5 = 5 + 9 + 7 = 7 + 5 + 9$
7. Teachers are expected to help pupils to understand the horizontal expanded motion algorism for addition.

a. $386 + 453 + 975 = (300 + 80 + 6) + (400 + 50 + 3)$
$+ (900 + 70 + 5)$
$= (300 + 400 + 900) + (80 + 50 + 70)$
$+ (6 + 3 + 5)$
$= 1{,}600 + 200 + 14$
$= 1{,}814$

B. **Subtraction.**

1. Teachers are expected to help pupils to understand the difference between subtraction as applied to sets and subtraction as a binary operation on a pair of numbers.
2. Pupils should understand subtraction as the inverse (opposite) operation of addition.
   a. Addition: $9 + 6 = 15$; 9 and 6 are the addends, 15 is the sum.
   b. Subtraction: $15 - 9 = \square$; 9 is the known addend, $\square$ stands for the unknown. $9 + \square = 15$ is the additive inverse. $9 + 6 = 15$.
   c. Pupils who understand and use the additive inverse method will not need to learn the standard subtraction combinations. They will use the standard combinations for addition.
3. Mathematical concepts to be applied to find the solution for the open sentence $314 - 87 = \square$

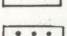
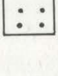
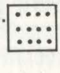
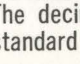
H	T	O
	10	14
2	$\cancel{0}$	$\cancel{10}$
$\cancel{3}$	$\cancel{1}$	$\cancel{4}$
	−8	7
2	2	7

(a) There is no whole number when added to 7 equals 4.
(b) The numeral that is the sum is renotated.
(c) **10 ones** is a replacement for 1 ten
(d) 10 ones + 4 ones = 14 ones
(e) **10 tens** is a replacement for 1 hundred
(f) The renotated sum is 2 H + 10T + 14 ones
(g) 7 + 7 = 14 ones (h) 8T+ 2T= 10T
(i) Write 2 in the hundreds place position.
(j) $227 + 87 = 314$

4. <u>If</u> and <u>then</u> open sentences may help pupils to improve their application of addition and subtraction
   a. If $9 + 8 = 17$, then $\square + 8 = 17$.
   a. If $9 + 8 = 17$, then $\square + 9 = 17$.
   c. If $17 - 9 = 8$, then $9 + \square = 17$.
5. There are no mathematical properties for subtraction on whole numbers.

VI. Mathematical Concepts and Properties Related to Operations on Whole Numbers. Multiplication and Division:

A. Multiplication: (Whole Numbers)

1. Multiplication is a binary operation on a pair of whole numbers to find a standard equivalent number. (It may be called product.)
2. Multiplication may be interpreted as rapid addition of equal valued addends (numbers).
3. $8 \times 9 = 72$ $(9+9+9+9+9+9+9+9= 72$
   (a) The first number of the pair (8×9) names the number of equal valued addends. When related to sets, it names the number of sets.
   (b) The second number of the pair $(8\times\underline{9})$, is one of the equal-valued addends. When related to sets it names the count of each set.
   (c) 72 is the product or standard equivalent.
4. Rectangular arrays are effective models for helping pupils to understand the 100 standard combinations for multiplication. (Whole numbers)
   a.  , $2 \times 3 = 6$ b.  , $3 \times 2 = 6$ c. , $3 \times 4 = 12$
   d.  , $4 \times 3 = 12$

5a. The decimal grid algorism is the most frequently used algorism to find the standard product for a pair of whole numbers.

Th	H	T	O
		3	4
		×5	8
		3	2
	2	4	
	2	0	
1	5		
1	9	7	2

$58 \times 34 = \square$
8 ones $\times$ 4 ones $=$ 32 ones $= 30 + 2 =$ 3T $+$ 2
8 ones $\times$ 3 tens $=$ 24T $=$ 20T $+$ 4T $=$ 2H $+$ 4T
5T $\times$ 4 ones $=$ 20T $=$ 2H $+$ 0T
5T $\times$ 3T $=$ 15H $=$ 10H $+$ 5H $=$ 1Th $+$ 5H
$58 \times 34 = 1{,}972$

5b. Shorter Algorism

Th	H	T	O
		3	4
		×5	8
	2	7	2
1	7	0	0
1	9	7	2

$8 \times 34 = (8 \times 4) + (8 \times 3T) = 272$

$5T \times 34 = (5T \times 4) + (5T \times 3T) = 1700$

5c. Multiplication, starting in the highest place-value positions

Th	H	T	O
		3	4
		×5	8
1	7	0	
	2	7	2
1	9	7	2

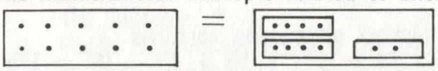
15 H $+$ 20T $=$ 10H $+$ 5H $+$ 2H $+$ 0T $=$ 1Th $+$ 7H $+$ 0T
8 ones x (3T $+$ 4 ones) $=$ 24T $+$ 32 ones $=$
20 T $+$ 4T $+$ 30 ones $+$ 2 ones $=$
2H $+$ 4T $+$ 3T $+$ 2 ones $=$
2H $+$ 7T $+$ 2 ones

6. Teachers should encourage pupils to think about each operational step to be taken to find the product for a pair of large whole numbers.
   a. $387 \times 954 = \square$ b. $405 \times 7{,}689 = \square$
7. Teachers are expected to help pupils to learn to use the open-sentence or horizontal algorism for multiplication.
   a. $58 \times 34 = \square$
   $= (50+8) \times (30+4)$
   $= (50 \times 30) + (50 \times 4) + (8 \times 30) + (8 \times 4)$
   $= 1{,}500 + 200 + 240 + 32$
   $= 1{,}972$
8. Teachers are expected to help pupils to understand that multiplication is an operation of a pair of factors to find the equivalent product.
   a. $8 \times 9 = 72$ 8 is one factor. 9 is the second factor. The product is 72.
   b. $58 \times 34 = 1972$ 58 is one factor. 34 is the second factor. The product is 1972.
9. The mathematical properties for multiplication (whole numbers), are:
   a. Identity property $1 \times 9 = 9$
   b. Closure property $8 \times 9 = 72$
   c. Commutative property $8 \times 9 = 9 \times 8$
   d. Associative property $7 \times 6 \times 5 = 5 \times 7 \times 6$
   e. Distributive over addition $5 \times (8+6) = (5 \times 8) + (5 \times 6)$
   f. Reciprocal property $8 \times 1/8 = 1$
10. Before starting computation to find the product for a pair of whole numbers, pupils should make an estimation of the product.
11. Pupils should verify the accuracy of a product by:
    (a) Repeating the multiplication operation
    or
    (b) Reversing the order of the pair of numbers:
    $24 \times 12 = \square$ $12 \times 24 = \square$
12. Pupils should understand that when there are three or more factors, they find the product for the first pair of factors and then use the product as a member of the next pair of factors: $8 \times 9 \times 7 = (8 \times 9) \times 7 = 72 \times 7 = 504$

B. Division: Whole Numbers

1. Pupils should discover and should understand that division is an inverse operation of multiplication.
   a. $6 \times 9 = 54 \quad 54 \div 9 = \square \quad \square \times 9 = 54$
   b. $9 \times 6 = 54 \quad 54 \div 6 = \square \quad \square \times 6 = 54$
2. When division is related to sets, the question to be answered is, "How many sub-sets having the same count may be obtained from a given set with a cardinal property?"

   $\boxed{\begin{matrix} XXXXX \\ XXXXX \end{matrix}} = \{XXXXX\}$ and $\{XXXXX\}$
3. Pupils should be prevented from thinking that division is an **into** operation. Pupils, when reading the open sentence, $24 \div 8 \ \square$, should not think, "How many times does 8 go into 24?" Instead, they should think, "How many 8's equal 24?" or "How many 8's are contained in 24?"
4. Pupils should use sets ot concrete materials or sets of symbols to discover the mathematical concepts related to uneven division.

   
   $= 10 \div 4 = 2$ Remainder 2
   $= 10 \div 4 = 2$ sets of 4 dots and 1 set of 2 dots

   $79 \div 22 = 3$ R13 $\qquad (3 \times 22) + 13 = 79$
5. Pupils may interpret division as an operation on a pair of whole numbers to find the unknown factor.

   $72 \div 8 = \square$ 8 is one factor $\square$ stands for the unknown factor
   a. Pupils should interpret the open sentence, "What number or factor times the known factor produces the known product 72?"
6. When the divisor or factor is named by a two-digit numeral or is named by a three-digit numeral, the division operation is more complexe
   a. $786 \div 4 = \square$ b. $\square \times 4 = 786$ c. $58{,}742 \div 308 = \square$
7. The decimal grid algorism is helpful when pupils are discovering that division is an inverse operation of multiplication.

Multiplication

$36 \times 6 = \square$

H	T	O
		6
	×3	6
1	8	
	3	6
2	1	6

Division

$216 \div 6 = \square$

		3	6
	H	T	O
6	~~2~~	1	6
		20	
		21	
		−18	
		~~3~~	30
			36
			36

8. An algorism may be used to illustrate division as an operation to subtract multiples of a known fact or from a given product.

$$\begin{array}{r l} & 29 \\ 25 \overline{)\ 725} & \\ -\ \underline{500} & (20 \times 25) = 500 \\ 225 & \\ -\ \underline{225} & (9 \times 25) = 225 \\ & (20 \times 25) + (9 \times 25) = 725 \end{array}$$

9. Teachers are expected to help pupils to discover and to understand why there are no mathematical properties for division as an operation on whole numbers.
10. If and Then open sentences may be used to help pupils to improve their understanding of division as a binary operation on a pair of whole numbers.
    a. If $7 \times 9 = 63$, then $63 \div 9 = \square$
    b. If $7 \times 9 = 63$, then $63 \div \square = 7$

c. If $29 \times 25 = 725$, then $725 \div 25 = \square$
d. If $29 \times 25 = 725$, then $725 \div \square = 29$

VII. Solving Story Problems and Open Mathematical Sentences

A. An important outcome derived from learning mathematics is the application of mathematical operations to find a solution to many personal problems and to many word or story problems.

B. The following generalizations merit examination by teachers.
1. Pupils may be handicapped in solving story problems when they have deficiencies in reading.
2. Pupils may be handicapped in solving story problems when they have deficiencies in computation.

C. Suggestions related to pupils' improvement in finding a solution for story problems.
1. Devote sufficient time for reading and analyzing each story problem.
2. Encourage pupils after they have read a story problem to give an appropriate answer to the following questions:
   a. What is the problem about?
   b. Who is the doer? Or, who are the doers.
   c. What information or fact is given?
   d. What is the primary question?
   e. What is the dependent question, if there is one in the problem?
   f. How may the problem be expressed in the form of an open mathematical sentence or an equation?
3. Encourage pupils, after they have translated or have expressed a story problem in the form of an open mathematical sentence or equation, to estimate what they think is a reasonable answer or solution.
4. Expand the opportunity for pupils to read and to analyze open mathematical sentences.

D. Enrichment activities related to solving story problems:
1. Arrange trips to collect information that pupils may use in writing problems.
2. Have pupils change open mathematical sentences to story problems.
3. Encourage pupils to prepare story problems related to experiences which may be found: (a) in the home; (b) in the school; (c) in the community.
4. Invite persons from business, industry, and the professions to tell how they use mathematics.
5. Prepare story problems which contain irrelevant facts or numbers related to a correct solution.
6. Have pupils to change story problems presented in the textbook by using different numbers.
7. Encourage pupils to bring to school story problems related to consumer buying.

VIII. Measures and Measurements

A. Ask pupils to think about their answers to the question, "What would a day be like without measure?"

B. Help pupils to discover and to understand the relation between a given magnitude and its measurement.

C. Provide experiences that will enable pupils to develop an understanding of:
1. a standard unit of measure;
2. a direct unit of measure;
3. a measurement;
4. units of measure for the measurement of length, area, volume, weight, time, speed, and temperature.

D. Provide materials and experiences which will assist pupils to understand the **Metric System** of measures.

E. Provide materials which will help pupils to understand the structure and function of a 24-hour clock or watch.

F. Prepare a chart to help pupils to understand the decimal structure of the **Metric System** of measures.

G. Provide experiences which will enable pupils to discover the quantitative relation between units of measure in the English, U.S.A. System, and units of measure in the **Metric System.**

IX. Teachers are expected to help pupils to learn the proper mathematical concept to be applied to each of the following terms and symbols:

Terms

1. Addend
2. Addition
3. Binary operation
4. A pair of numbers
5. Decimal numeration system
6. Decimal notation
7. Division (whole numbers)
8. Divisor
9. Expanded notation
10. Equivalent sets
11. Factor
12. Identical sets
13. Known factor
14. Multiplication (whole numbers)
15. Multiple
16. Multiplier
17. Number
18. Numeral
19. Notation
20. Product (whole numbers)
21. Quotient
22. Remainder (uneven division)
23. Set
24. Sub-set
25. Sum
26. Whole number

Symbols

1. "$=$" "$\neq$"
2. "$\times$" "$-$" "$+$" "$\div$" "/ "
3. "{ }"
4. $A < B$ $A > B$ $A = B$
5. $a \times b = c$, If a and $b \neq 0$
6. $a + b = c$
7. $c - b = a$
8. $c \div b = a$, If $b \neq 0$ and $a \times b = c$
9. $(a \times b) + r = c$, a and $b \neq 0$
10. $c \div b = a + r$, $b \neq$ zero
11. "( )" "[ ]"

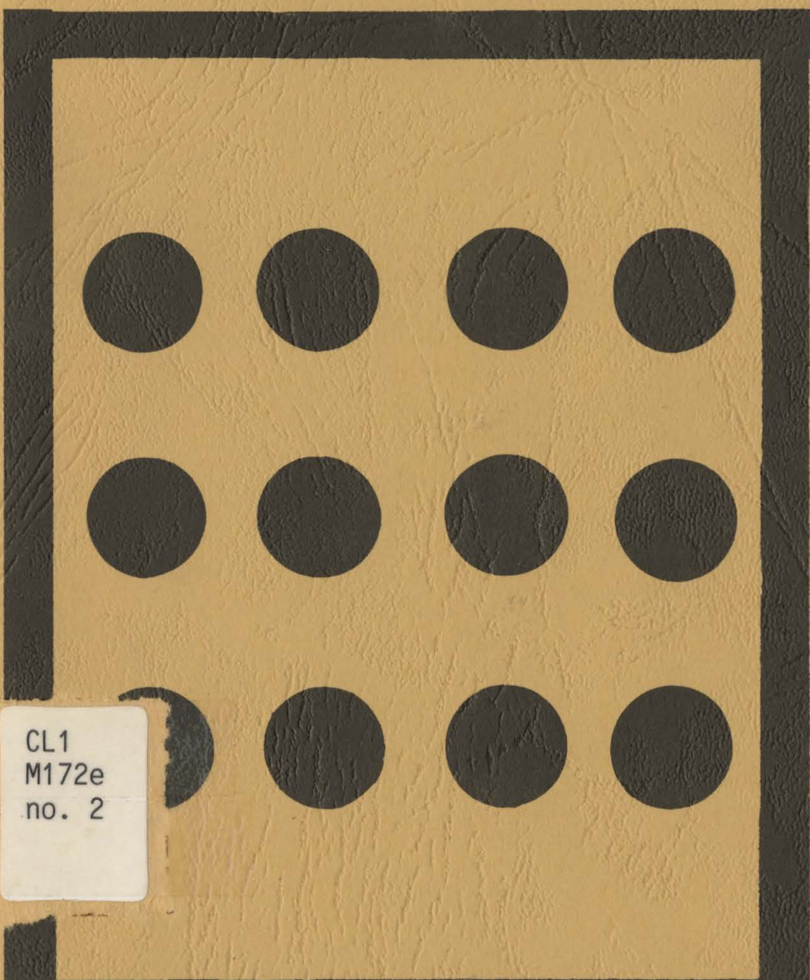
THE PIEDMONT ASSOCIATION
FOR
SCHOOL STUDIES AND SERVICES

GUIDE NUMBER TWO

elementary mathematics

selected numeration systems

E. T. McSwain



12
TEN

1100
TWO

14
EIGHT

10
TWELVE

XII

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for

The Piedmont Association for School Studies and Services
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Systems of Numeration

This guide is related to the five numeration systems which teachers are expected to help pupils to understand and to apply.

The **Hindu-Arabic Numeration System** ... now called the decimal system, is the most widely used system.

The **Binary Numeration System** .... base two, is important for the reason that it is the structured base for electronic computers.

The **Octal Numeration System** .... base eight, is important for the reason that numerals in the binary system can be translated easily to numerals in base eight.

The **Duo-decimal Numeration System**..... has many applications when expressing certain measures and measurements, such as: twelve inches, twelve hours, one dozen, twelve months.

The **Roman Numeration System**.......... has been presented for the reason that Roman Numerals are used frequently to notate pages in books and for recording important dates.

Purposes for Teaching These Five Numeration Systems

The reasons for including these numeration systems in the elementary mathematics curriculum are:

1. To present mathematical concepts and experiences that may assist pupils to improve their understanding of the mathematical concepts, structure, terms, and operations related to the decimal numeration system.
2. To encourage pupils to improve their understanding of other numeration systems which are encountered in contemporary society.
3. To motivate pupils to construct models that illustrate the structure of these numeration systems.
4. To provide pupils the opportunity to learn to convert numerals in one system of numeration to numerals in other numeration systems.
5. To assist pupils to experience interest from operations on numbers in different systems of numeration.

Since numbers are abstract mathematical concepts that have meaning and function only in the mind, it is imperative that a system of numeration be structured for recording and communicating number ideas. The cardinality property or count of each set is independent of the numeral used to name the number.

It is important that pupils discover and understand the difference between a numeration system and a number system. A numeration system is a mathematical way for naming or notating numbers by symbols or digits controlled by established order and rules. A number system involves broader mathematical concepts and properties. The three basic requirements of a number system are:

1. a set of ordered numbers.
2. two binary operations; addition and multiplication and their inverse operations.
3. established properties controlling the binary operations: addition and multiplication.

The principles of numeration cannot be understood correctly by pupils until they understand the mathematical difference between the meaning of numbers and the function of numerals. Numbers are mathematical concepts (ideas) assigned to standard sets. Numerals are symbols used to express or to notate number concepts.

Characteristics of Numeration Systems

Teachers are expected to help pupils to discover and to understand the mathematical characteristics which are similar for the decimal, binary, octal, and duo-decimal numeration systems. These characteristics are:

1. Each numeration system has a defined base.
   (a) The base for the Decimal System is Ten.
   (b) The base for the Binary System is Two.
   (c) The base for the Octal System is Eight.
   (d) The base for the Duo-decimal System is Twelve.

2. Each numeration system has a set of standard numerals called digits. The number of standard digits equal the cardinality property or count of the base.
   (a) The standard digits for the Decimal System are:
       0,1,2,3,4,5,6,7,8,9
   (b) The standard digits for the Binary System are: 0,1
   (c) The standard digits for the Octal System are:
       0,1,2,3,4,5,6,7.
   (d) The standard digits for the Duo-decimal System are:
       0,1,2,3,4,5,6,7,8,9,T,E,
3. Each numeration system has an ordered place-value or positional structure. The first place-value position has the value of one. The second place-value position to the left of the first position is assigned the value of the base. Place-value positions, in ordered sequence, to the left of the second or base-value position have a value that is a power of the base.

$B \times B \times B \times B$ $\quad$ $B \times B \times B$ $\quad$ $B \times B$ $\quad$ Base $\quad$ One

Place-Value Positions for the Selected Numeration Systems

The chart shown below is an illustration of the ordered sequence of the place-value positions and the precise digital numeral that is the digital name for each place-value position.

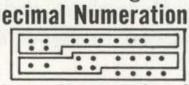
Place-Value Structure for the Selected Numeration Systems

Decimal System Base (10^1) Digits: 0,1,2,3,4, 5,6,7,8,9	(10×1000) or 10^4 10,000 Ten Thousand	$(10 \times 10 \times 10)$ or 10^3 1000 Thousand	(10×10) or 10^2 100 Hundred	10^1 10 Ten	10^0 1 One
Binary System Base Two (10^1) Digits: 0,1	(10×1000) or 10^{100} 10,000 Sixteen	$(10 \times 10 \times 10)$ or 10^{11} 1000 Eight	(10×10) or 10^{10} 100 Four	10^1 10 Two	10^0 1 One
Octal System Base Eight (10^1) Digits: 0,1,2,3, 4,5,6,7	(10×1000) or 10^4 10,000 Four Thousand Ninety-six	(10×100) or 10^3 1000 Five Hundred Twelve	(10×10) or 10^2 100 Sixty-four	10^1 10 Eight	10^0 1 One
Duo-decimal System Base Twelve (10^1) Digits: 0,1,2,3,4, 5,6,7,8,9 T,E,	10(1000) or 10^4 10,000 Twenty Thousand Seven Hundred Thirty-six	(10×100) or 10^3 One Thousand Seven Hundred Twenty-eight	(10×10) or 10^2 One Hundred Forty-four	10^1 10 Twelve	10^0 1 One

Numbers and Numerals for the Selected Numeration Systems

The cardinality property (the count) and mathematical numbers are the same for standard sets and their relation to other sets. Number property and numbers do not belong to standard sets; they are mental concepts or abstractions. Digital numerals are used to name or to notate numbers. The notation of the numerals is dependent upon the base and the ordered place-value structure of each numeration system. The method or process used for grouping elements in a standard set to determine the digital numeral that is the name for the number is illustrated in the charts given below.

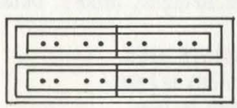
1. **The Decimal Numeration System**



= two subsets of ten and one subset of four =
2 tens + 4 ones = 24

2. **The Binary Numeration System**

A.



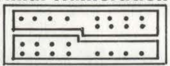
One subset of two twos form
one subset of one four (100)
Two subsets of four each form
one subset of one eight (1000)
= Two subsets of eight each form
one subset of one sixteen (1000)
The numeral that names the cardinality count of set A is 1100

$$\begin{array}{ccccc} 10^{100} & 10^{11} & 10^{10} & 10^{1} & 10^{0} \\ 1 + & 0 + & 0 + & 0 + & 0 \end{array} = 10000 \text{ Two}$$

3. **The Octal Numeration System**

 $= \text{Three subsets} = 3 \text{ eights} + \text{no ones} = \begin{array}{cc} 10^{1} & 10^{0} \\ 3 & 0 \end{array}$

4. **The Duo-decimal Numeration System**

 $= \text{Two subsets} = 2 \text{ twelves} + 0 \text{ ones} = \begin{array}{cc} 10^{1} & 10^{0} \\ 2 & 0 \end{array}$

Pupils when motivated to experiment with sets of objects, such as checkers, bottlecaps, toothpicks, etc., may soon discover and understand the mathematical meaning of numbers and their numerals.

Numbers and Numerals Chart

The chart given below can be helpful to pupils in understanding notation of some numbers in each of the four numeration systems.

Number	Decimal Base Ten	Binary Base Two	Octal Base Eight	Duo-decimal Base Twelve
	$(10)^1$	$(10)^1$	$(10)^1$	$(10)^1$
One	1	1 two	1 eight	1 twelve
Two	2	10 two	2 eight	2 twelve
Three	3	11 two	3 eight	3 twelve
Four	4	100 two	4 eight	4 twelve
Five	5	101 two	5 eight	5 twelve
Six	6	110 two	6 eight	6 twelve
Seven	7	111 two	7 eight	7 twelve
Eight	8	1000 two	10 eight	8 twelve
Nine	9	1001 two	11 eight	9 twelve
Ten	10	1010 two	12 eight	T twelve
Eleven	11	1011 two	13 eight	E twelve
Twelve	12	1100 two	14 eight	10 twelve
Fifteen	15	1111 two	17 eight	13 twelve
Twenty-four	24	11000 two	30 eight	20 twelve
Seventy-Seven	77	1001101 two	115 eight	65 twelve
One Hundred	100	1100100 two	144 eight	84 twelve
One Hundred Twenty-four	124	1111100 two	174 eight	T4 twelve

Number is the cardinality property or count of a given set. Numbers are the same for all NUMERATION SYSTEMS.

Numerals, names for numbers, are dependent upon the place-value structure of the NUMERATION SYSTEMS.

Expanded Notation

Expanded notation may be used for each of the selected numeration systems. Each place-value position is expressed by a numeral that is the product of the face value digit and the base used as a factor.

1. In the **Decimal System:**

$$465 = 4(10\times10) + 6(10) + 5$$
$$= 4(100) + 6(10) + 5$$
$$\text{Exponential notation} = 4(10)^2 + 5(10)^1 + 5$$

2. In the **Binary System:**

$$1101 = 1(10\times10\times10) + (10\times10) + 0(10) + 1$$
$$= 1(1000) + 1(100) + 0(10) + 1$$
$$\text{Exponential notation} = 1(10)^{11} + 1(10)^{10} + 0(10)^1 + 1$$

3. In the **Octal System:**

$$3345 = 3(10\times10\times10) + 3(10\times10) + 4(10) + 5$$
$$= 3(1000) + 3(100) + 4(10) + 5$$
$$\text{Exponential notation} = 3(10)^3 + 3(10)^2 + 4(10)^1 + 5$$

4. In the **Duo-decimal System:**

$$3345 = 3(10\times10\times10) + 3(10\times10) + 4(10) + 5$$
$$= 3(1000) + 3(100) + 4(10) + 5$$
$$\text{Exponential notation} = 3(10)^3 + 3(10)^2 + 4(10)^1 + 5$$

Teachers are expected to help pupils to express numerals in the selected notation systems in the form of expanded notation and also in the form of exponential notation. As pupils learn to express numbers by their numerals in the selected numeration systems they may be encouraged to apply these forms of notation when executing an operation (addition, subtraction, multiplication, division) on a pair of whole numbers.

1. **Addition:**

$$\text{(Base Ten) } 346 + 253 = 3(100) + 4(10)+6 + 2(100) + 5(10) + 3$$
$$= (300 + 40 + 6) + (200 + 50 + 3)$$
$$= (300 + 200) + (40+50) + (6+3)$$
$$= 500 + 90 + 9 = 599$$
$$\text{Expanded Notation} = 3(10)^2 + 4(10)+6 + 2(10)^2 + 5(10)^1 + 3$$

2. **Addition:**

$$\text{(Base Two) } 111 + 101 = [1(100) + 1(10) + 1] + [1(100) + 0(10)+1]$$
$$= 1(100)+1(100) + 1(10) + 0(10) + (1+1)$$
$$= 1(1000) + 1(10) + 1(10) = 1100$$

Teachers and pupils may prepare illustrations for expanded and exponential notation as applied to subtraction and multiplication in other systems of numeration.

Exponents in the Numeration Systems

Pupils should discover and should understand that the number for the base in each of the numeration systems is called a factor. The superscript written to the right of the base is called an exponent. It identifies the number of times the base has been used as a factor.

1. Base ten 10^3 equals $10\times10\times10$. The simplest numeral is 1000_{ten}.
2. Base two 10^{11} equals $10\times10\times10$. The simplest numeral is 1000_{two}.
3. Base eight 10^3 equals $10\times10\times10$. The simplest numeral is 1000_{eight}.
4. Base twelve 10^3 equals $10\times10\times10$. The simplest numeral is 1000_{twelve}.

The exponent expresses the power of a base.

$10^3{}_{ten}$ is the exponent form or $10_{ten} \times 10_{ten} \times 10_{ten}$

$10^{11}{}_{two}$ is the exponent form or $10_{two} \times 10_{two} \times 10_{two}$

$10^3{}_{eight}$ is the exponent form or $10_{eight} \times 10_{eight} \times 10_{eight}$

Importance of the Binary System of Numeration

Pupils, when they reach adult age, will enter employment in business, industry, professions, or governmental agencies and they will encounter results obtained from electronic computers. The structure and operational processes of the electronic computer are based on the Binary Numeration System, or base two. The problem encountered by large digital numerals in base two is offset by the speed of electricity. An electric circuit has only two responses. It is either open (on), closed (off). A one-to-one correspondence can be established between the Binary Numeration System and the open or closed circuit of electricity. In the electronic computer a closed circuit or a light represents on, and no light

represents an off circuit. There are only two responses in an electric circuit, closed (on) and open (off).

With the assistance of the teachers of science or industrial arts, pupils may experience interest and motivation by using an electric circuit and a string of light bulbs to build a model of the binary system of notation. They may extend an invitation to engineers or business executives to visit their classroom to present the values of the electronic computer. In an electronic computer, the numeral 1101 is notated by lights as: one (first light on), two (second light off), one four (third light on) and one eight (fourth light on) and is read from left to right, 1101_{two}.

The Duo-decimal Numeration System

The Duo-decimal Numeration System has a base that has the cardinality property of twelve and an ordered sequence of place-value positions with values of twelve and powers of twelve.

(Twelve)^4	Twelve $\times$ Twelve $\times$ Twelve	Twelve $\times$ Twelve	Twelve	One
$(10)^4$	$(10)^5$	$(10)^2$	$(10)^1$	$(10)^0$
10,000	1000	100	10	1

Since there are only ten standard digits in the Decimal System of Numeration, (0,1,2,3,4,5,6,7,8,9) two additional symbols (T, E,) were added to complete the set of standard digits in the Duo-decimal Numeration System. The set of standard digits are: 0,1,2,3,4,5,6,7,8,9,T,E. The Duo-decimal numeral 8T4E is interpreted to express:

$$8(\text{twelve}^3) + \text{ten}(\text{twelve}^2) + 4(\text{twelve})^1 + \text{eleven ones}$$

Pupils should be encouraged to engage in library research to find ways the Duo-decimal Numeration System is used to express and to record some measurement in business or in industry.

Conversion of Numerals in the Decimal System of Numeration to Corresponding Numerals in Other Numeration Systems

When converting decimal numbers to numerals in the Binary Numeration System, pupils, from their study of the Decimal and Binary Numeration Systems, should understand the value of each base and the order and value of each place-value position. The Decimal numeral, 24, names two tens and four ones. This numeral was obtained by thinking that a set of **twenty-four ones** was arranged into **two subsets** of **ten ones** and one subset of **four ones.**

Two algorisms may be used when converting a Decimal numeral to a corresponding numeral in the Binary Numeration System; the Division Algorism and the Factoring Algorism.

1. **The Division Algorism:** arranging the cardinality count of the number into subsets of two and powers of two.

	A set of **twenty-four ones** contains
2 / 2 4 R 0	**twelve** subsets of **two ones** and no remainder.
2 / 1 2 R 0	**Twelve** subsets of **two** form **six** subsets of **two twos** or **four** and no remainder.
2 / 6 R 0	**Six** subsets of **four** form **three** subsets of **eight** and no remainder.
2 / 3 R 1	**Three** subsets of **eights** (two$\times$two$\times$two)
1	forms **one** subset of **sixteen** with a remainder, **one** subset of **one eight.**

2. **The Factor Algorism:** requires thinking of a power of two that equals or is closer to 24. Two raised to the four power is 16. When 16 is subtracted from 24 the remainder is 8.

$$\begin{array}{r} 24 \\ -\underline{16} \\ 8 \end{array} = 1(10)^{100}$$

Two raised to the third power is 8. Eight subtracted from 8 leaves no units remaining. The Binary numeral for 24_{ten} is 11000_{two}.

The Algorism to be used when converting a numeral in the Binary Numeration System to a corresponding Decimal Numeral requires that the value of each position in the Binary numeral be expressed by a numeral in the Decimal System.

$1110_{two} = 1(8) + 1(4) + 1(2) + 0(1) = 8 + 4 + 2 + 0 = 14_{ten}$
$11000_{two} = 1(16) + 1(8) + 0(4) + 0(2) + 0(1) = 16 + 8 + 0 + 0 + 0 = 24_{tens}$

Pupils should be given practice in converting numerals in the Binary Numeration System to corresponding numerals in the Decimal System. They may verify the results by then converting the Decimal numerals to corresponding numerals in the Binary System.

1. Converting decimal numerals to corresponding numerals in the Octal or Duo-decimal Numeration System may be done by applying the division Algorism of the Factor Algorism provided the value of the base in the Octal or in the Duo-decimal Numeration System is used.

Converting 421_{ten} to __________ eight

$8 \overline{/4\ 2\ 1_{ten}}$ R 5
$8 \overline{/5\ 2}$ R 4
6

645_{eight}

$[6(8\times8) + 4(8) + 5]$ 645_{eight}

Converting 421_{ten} to __________ twelve

$12 \overline{/4\ 2\ 1_{ten}}$ R 1
$12 \overline{/35}$ R E
2

$2E1_{twelve}$

$2(12\times12) + E(12) + 1$ $2E1_{twelve}$

Factor Algorism

421_{ten}
$-\ \underline{384}\ (6\times64) = 6(10)^2$
37
$-\ \underline{32}\ (4\times8) = 4(10)^1$
$5 = 5(10)^0$

645_{eight}

421_{ten}
$-\ \underline{288}\ \ 2(12\times12)$
133
$-\ \underline{132}$
1 $2E1_{twelve}$ $2E1_{twelve}$

The Algorism to be used when converting a numeral in base eight or in base twelve is to express each place value position by a decimal numeral and find the sum of the partial products.

645_{eight} to __________ ten
$6(8\times8) + 4(8) + 5$
$6(64) + 32 + 5$
$384 + 32 + 5$
421_{ten}

$2E\ 1_{twelve}$ to __________ ten
$2(12\times12) + 11(12) + 1$
$288 + 132 + 1$
421_{ten}

The Roman Numeration System

This numeration system was used before the invention of the printing press. The standard digits or numerals used in the Roman System of Numeration are: I, V, X, L, C, D, and M. The symbol "I" is the name for one, the symbol "V" is the name for five, the symbol "X" is the name for ten, the symbol "L" is the name for fifty, the symbol "C" is the name for one hundred, the symbol "D" is the name for five hundred and the symbol "M" is the name for one thousand. The symbol —(a bar) drawn over a standard digit indicated the value expressed by the digit is to be multiplied by one thousand ($\overline{L} = M\times L$ = five thousand.) The system has a decimal characteristic but does not have an ordered place-value structure. The additive principle is used. The number value of any numeral is equal to the sum of place-value of the standard numerals. CCLXXIII = 100 + 100 + 50 + 10 + 10 + 3 = 273. Subtraction principle may be applied in order to reduce the number of repetitions of a symbol. IIII = IV, XL = L—X, IX = Nine, CD = D—C = four hundred.

The absence of an ordered place-value structure has caused complexity when applying the operations of addition and multiplication on a pair of large numbers and their Roman Numerals.

Mathematical Concepts Related to Addition and Subtraction—Different Systems of Numeration

The important mathematical concepts (ideas) that pupils should understand are:

1. Cardinality property (a count) is not an element of the set; it is a mathematical concept assigned to a given set of elements.
2. A number is not an element contained in a given set; it is a mathematical and mental concept assigned to a set of elements.
3. Numbers are named by numerals dependent upon the numeration system controlling the numeral.

X	X	X	X	X	X	X	X
X	X	X	X	X	X	X	X

Numerals for the Number
1000 Base Two; 20 Base Eight; 16 Base Ten; 14 Base Twelve

Addition—A binary operation on a pair of numbers and their numerals to find the standard or unique equivalent.

1. **Binary Numeration System.**

$$\begin{array}{rl} 111_{two} & \\ +\ \underline{101_{two}} & 111_{two} + 101_{two} = \square \\ 10 & 1 \text{ one} + 1 \text{ one} = \text{two ones} = 1 \text{ two} + 0 \text{ ones} \\ 100 & 1 \text{ two} + 1 \text{ two} = \text{two twos} = 1 \text{ four} + 0 \text{ twos} \\ \underline{1100} & \underline{1 \text{ four} + 1 \text{ four} + 1 \text{ four} = 1 \text{ eight} + 1 \text{ four}} \\ 1100 & 1 \text{ eight} + 1 \text{ four} + 0 \text{ twos} + 0 \text{ ones} \end{array}$$

2. **Octal Numeration System.**

$$\begin{array}{rl} 274_{eight} & 4 \text{ ones} + 5 \text{ ones} = 1 \text{ eight and } 1 \text{ one} \\ +\ \underline{765_{eight}} & \\ 11_{eight} & 7 \text{ eights} + 6 \text{ eights} + 1 \text{ eight} = 1 \text{ eight} \times \text{eight} + 6 \text{ eights} \\ 16\ _{eight} & 2(\text{eight})^2 + 7(\text{eight})^2 = 1(\text{eight})^3 + 1(\text{eight})^2 \\ \underline{12} & \\ 1261_{eight} & = 1\,(\text{eight}^3) + 2(\text{eight}^2) + 6(\text{eight})^1 + 1(\text{one}) \end{array}$$

3. **Duo-decimal Numeration System.**

$$\begin{array}{rl} 386_{twelve} & \\ +\ \underline{947_{twelve}} & \\ 11 & 6 \text{ ones} + 7 \text{ ones} = 1 \text{ twelve} + 1 \text{ one} \\ 11 & 8(\text{twelve}) + 4(\text{twelve}) + 1(\text{twelve}) = 1(\text{twelve}^2) + 1(\text{twelve}) \\ \underline{11} & 3(\text{twelve})^2 + 9(\text{twelve})^2 + 1(\text{twelve})^2 = 1(\text{twelve})^3 + 1(\text{twelve})^2 \\ 1111_{twelve} & = 1(10)^3\text{twelve} + 1(10)^2\text{twelve} + 1(10)^1\text{twelve} + 1 \end{array}$$

Subtraction—the inverse operation of addition.

1. **Binary Numeration System.**

$$\begin{array}{rl} 1100_{two} & 101_{two} + \square_{two} = 1100_{two} \\ -\ \underline{101_{two}} & 101_{two} + 111_{two} = 1100_{two} \\ N & \end{array}$$

2. **Octal Numeration System.**

$$\begin{array}{rl} 1261_{eight} & 765_{eight} + \square_{eight} = 1261_{eight} \\ -\ \underline{765_{eight}} & 765_{eight} + 274_{eight} = 1261_{eight} \\ N & \end{array}$$

3. **Duo-decimal Numeration System.**

$$\begin{array}{rl} 1111_{twelve} & \\ -\ \underline{947_{twelve}} & \\ N & 947_{twelve} + \square_{twelve} = 1111_{twelve} \end{array}$$

$$\begin{array}{r} 1\ 1\ 1\ 1_{twelve} \\ -\ \underline{9\ 4\ 7_{twelve}} \\ 3\ 8\ 6_{twelve} \end{array} \qquad 947_{twelve} + 386_{twelve} = 1111_{twelve}$$

Multiplication — Selected Numeration Systems

A. Important Mathematical Concepts

It is the opportunity of teachers in the intermediate grades to offer instruction, experiences and material to enable pupils to discover and to develop a foundational understanding of the following mathematical concepts:

1. Multiplication, in each numeration system is a binary operation of a pair of numbers to find the number that is the unique, standard equivalent or product.
2. The numbers in the different numeration systems are the same; only the numerals are different.
3. There is a set of standard multiplication combinations for each numeration system.
4. Two grid methods or algorisms may be used to find the standard equivalent or product for any given pair of numbers—
   (a) The typical method when using the vertical grid algorism enables pupils to start multiplication by beginning with the pair of numbers in the ones place-value position and continuing to the next higher place-value positions. (Refer to c-1)
   (b) A second vertical grid method, when used, will enable pupils to comprehend the relation between multiplication and division. This method requires pupils to begin multiplication by first finding the partial product for the pair of numbers in the highest place-value position and then moving to pairs in the

next lower place-value positions.

(c) 1. $8_{ten} \times 74_{ten} = N$

H	T	O
	7	4
		×8
	3	2
5	6	
5	9	2

5. Pupils, after reading the open sentence that requires multiplication for a solution, should be encouraged to make a mental estimate of a reasonable solution.
6. Pupils should be encouraged to discover and to understand:
   (a) That multiplication is an operation to find by rapid addition the standard equivalent or product for any given pair of whole numbers.
   (b) That each partial product expresses the unique, standard equivalent for the pair of numbers in each place-value position.
   (c) That the total sum is the equivalent for the sum of the partial products in the respective place-value positions.
7. Pupils should be expected to discover and to understand the mathematical reason why division is the opposite operation for multiplication:
   (a) Multiplication is a process of rapid addition (Rapid addition)
   (b) Division is an opposite operation of multiplication. (Rapid subtraction)
   (c)

H	T	O
	7	4
		×8
5	6	
	3	2
5	9	2

$8 \times 7T = 5H + 6T$
$8 \times 4 = 3T + 2$
$5H + 9T + 2$

	H	T	O
		7	4
8	~~5~~	9	2
		50	
		59	
		56	
		~~3~~	30
			32
			—32

$59T \div 8 = 7TR3T$
$32 \text{ ones} \div 8 = 4$

B. Multiplication and the Opposite Operation, Division

1. Base Ten (10)[1] (a) $96 \times 7 = \square$ (b) $72 \div 7 = \square$

(a)

H	T	O
		7
	×9	6
6	3	
	4	2
6	7	2

$9T \times 7 = 6H + 3T$
$6 \times 7 = 4T + 2$
$6H + 7T + 2$

$96 \times 7 = 672$

(b)

	H	T	O
		9	6
7	6	7	2
		60	
		67	
		—63	
		~~4~~	40
			42
			—42

$6H = 60T$
$60T + 7T = 67T$
$9T \times 7 = 63T$
$4T = 40 \text{ ones}$
$40 \text{ ones} + 2 \text{ ones} = 42 \text{ ones}$
$6 \times 7 = 42 \text{ ones}$

$672 \div 7 = 96$

(c) Base Ten $(10)^1$ $46 \times 38 = \square$

Th	H	T	O
		3	8
		×4	6
1	2		
	3	2	
	1	8	
		4	8
1	7	4	8

$4T \times 3T = 12H = 10H + 2H = 1Th + 2H$
$4T \times 8 = 32T = 30T + 2T = 3H + 2T$
$6 \times 3T = 18T = 10T + 8T = 1H + 8T$
$6 \times 8 = 48\text{ones} = 40 + 8 = 4T + 8$
$4T \times (3T + 8) + 6\,(3T + 8) = 1748$

$46_{\text{ten}} \times 38_{\text{ten}} = 1748_{\text{ten}}$

(d) Division, Base Ten $1748 \div 38 = \square$ or $\square \times 38 = 1748$
$46 \times 38 = 1748$

	Th	H	T	O
38	×	7	4	8
		10		
		~~17~~	170	
			174	
			152	
			~~22~~	220
				228
				228

1Th does not contain a set of 38Th
$1Th = 10H$ $\quad 10H + 7H = 17H$
10H does not contain 38H
$17H = 170T$ $\quad 170T + 4T = 174T$
$4 \times 38T = 152T$ $\quad 174T - 152T = 22T$
$22T = 220$ ones $\quad 220 + 8 = 228$
$6 \times 38 = 228$

2. **Base Two** $(10)^1$ Multiplication
(a) $11_{\text{two}} \times 11_{\text{two}} = \square$

10^{11}	10^{10}	10^1	10^0
		1	1
		×1	1
	1		
		1	
		1	
			1
1	0	0	1

$1\,(10)^1 \times 1\,(10)^1 = 1\,(10)^{10}$
$1\,(10)^1 \times 1\,(10)^0 = 1\,(10)^1$
$1\,(10)^0 \times 1\,(10)^1 = 1\,(10)^1$
$1\,(10)^0 \times 1\,(10)^0 = 1\,(10)^0$
$= 1\,(10)^{11} + 0(10)^{10} + 0(10)^1 + 1(10)^0$

$11_{\text{two}} \times 11_{\text{two}} = 1001_{\text{two}}$

Base Two $(10)^1$ - Division (a) $1001_{\text{two}} \div 11_{\text{two}} = \square$

	10^{11}	10^{10}	10^1	10^0
11_{two}	~~1~~	0	0	1
		~~10~~		
			100	
			100	
			−11	
			~~1~~	10
				11
				−11

$1\,(10)^{11}$ does not contain 11_{two}
$1\,(10)^{11} = 10(10)^{10}$
$10(10)^{10}$ does not contain 11_{two}
$10\,(10)^{10} = 100(10)$
$100(10)^1$ contains 1 set of 11_{two} with R $(10)^1$
$1\,(10)^1 = 10(10)^0$
$10(10)^0 + 1\,(10)^0 = 11(10)^0$
$11(10)^0$ contains 1 set of 11_{two} No Remainder

$1001_{\text{two}} \div 11_{\text{two}} = 11_{\text{two}}$

Suggestions:

(1) It is suggested that division in Base Two be restricted to a two place-value divisor until pupils understand the operation.
(2) Rapid achievers may experiment with division Base Two, when dividend and divisor are larger numbers.
(3) Require pupils to verify accuracy of division operation by multiplying the divisor by the quotient.

3. **Base Eight**—Multiplication and the Opposite Operation—Division

(a) Multiplication

$64_{eight} \times 7_{eight} = \square_{eight}$

$(10)^2$	$(10)^1$	$(10)^0$
		7
	$\times 6$	4
5	2	
	3	4
5	5	4

(b) Division—Opposite Operation

$554 \div 7_{eight} = \square_{eight}$

$6(10)^1 \times 7(10)^0 = 52(10)^1 = 5(10)^2 + 2(10)^1$

$4(10)^0 \times 7(10)^0 = 34(10)^0 = 3(10)^1 + 4(10)^0$

$[6(10)^1 + 4(10)^0] \times 7(10)^0 = 5(10)^2 + 5(10)^1 + 4(10)^2$

(b) **Division**—Base Eight

$554_{eight} \div 7_{eight} = \square$ OR $\square \times 7_{eight} = 554_{eight}$

	10^2	10^1	10^0
7_{eight}	$\cancel{5}$	5	4
		50	
		55	
		-52	
		$\cancel{3}$	
			30
			34
			34

$5(10)^2 = 50(10)^1$

$50(10)^1 + 5(10)^1 = 55(10)^1$

$6(10)^1 \times 7(10)^0 = 52(10)^1$

$3(10)^1 = 30(10)^0$

$30(10)^0 + 4(10)^0 = 34(10)^0$

$4(10)^0 \times 7(10) = 34(10)^0$

$554_{eight} \div 7_{eight} = 64_{eight}$

$64_{eight} \times 7_{eight} = 554_{eight}$

Suggestions:

(1) It is suggested that division in Base Eight be restricted to a one place-value divisor until pupils understand the operation.

(2) Require pupils to verify accuracy of the division operation by multiplying the divisor by obtained quotient.

(3) Rapid achievers may experiment with division (Base Eight) when dividend and divisor are larger numbers.

3. **Base Twelve**

It is recommended that multiplication and division as operations on numbers named in Base Twelve be postponed until the junior high school.

Mathematical Properties for Addition and Multiplication in the Selected Numeration Systems

A. Teachers in the intermediate grades are expected to help pupils to discover and to understand the mathematical properties for addition in Base Ten, Base Two, and Base Eight. (Base Twelve may be postponed until the junior high school.)
   1. Identify Property
   2. Closure Property
   3. Commutative Property
   4. Associative Property

B. Teachers in the intermediate grades are expected to help pupils to discover and to understand the mathematical properties for multiplication in Base Ten, Base Two, Base Eight. (Base Twelve may be postponed until the junior high school.)
   1. Identify property
   2. Closure property
   3. Commutative property
   4. Associative property
   5. Distribution over addition property

Samples of open mathematical sentences to be solved after pupils have studied the basic binary operations and their inverse operations on numbers named in Base Ten, Base Two, and Base Eight.

A. Addition

1. $76_{ten} + 84_{ten} = \square_{ten}$
2. $876_{ten} + 904_{ten} = \square_{ten}$
3. $11_{two} + 11_{two} = \square_{two}$
4. $111_{two} + 101_{two} = \square_{two}$
5. $37_{eight} + 45_{eight} = \square_{eight}$

B. Subtraction

1. $160_{ten} - 84_{ten} = \square_{ten}$
2. $1780_{ten} - 876_{ten} = \square_{ten}$
3. $110_{two} - 11_{two} = \square_{two}$
4. $1100_{two} - 111_{two} = \square_{two}$
5. $104_{eight} - 45_{eight} = \square_{eight}$

C. Multiplication

1. $96_{ten} \times 7_{ten} = \square_{ten}$
2. $46_{ten} \times 38_{ten} = \square_{ten}$
3. $11_{two} \times 11_{two} = \square_{two}$
4. $111_{two} \times 11_{two} = \square_{two}$
5. $64_{eight} \times 7_{eight} = \square_{eight}$

D. Division

1. $672_{ten} \div 7_{ten} = \square_{ten}$
2. $1748_{ten} \div 38_{ten} = \square_{ten}$
3. $1001_{two} \div 11_{two} = \square_{two}$
4. $10101_{two} \div 11_{two} = \square_{two}$
5. $554_{eight} \div 7_{eight} = \square_{eight}$

THE PIEDMONT ASSOCIATION
FOR
SCHOOL STUDIES AND SERVICES

GUIDE NUMBER THREE

elementary mathematics

mathematical concepts, operations, and properties related to the system of rational numbers

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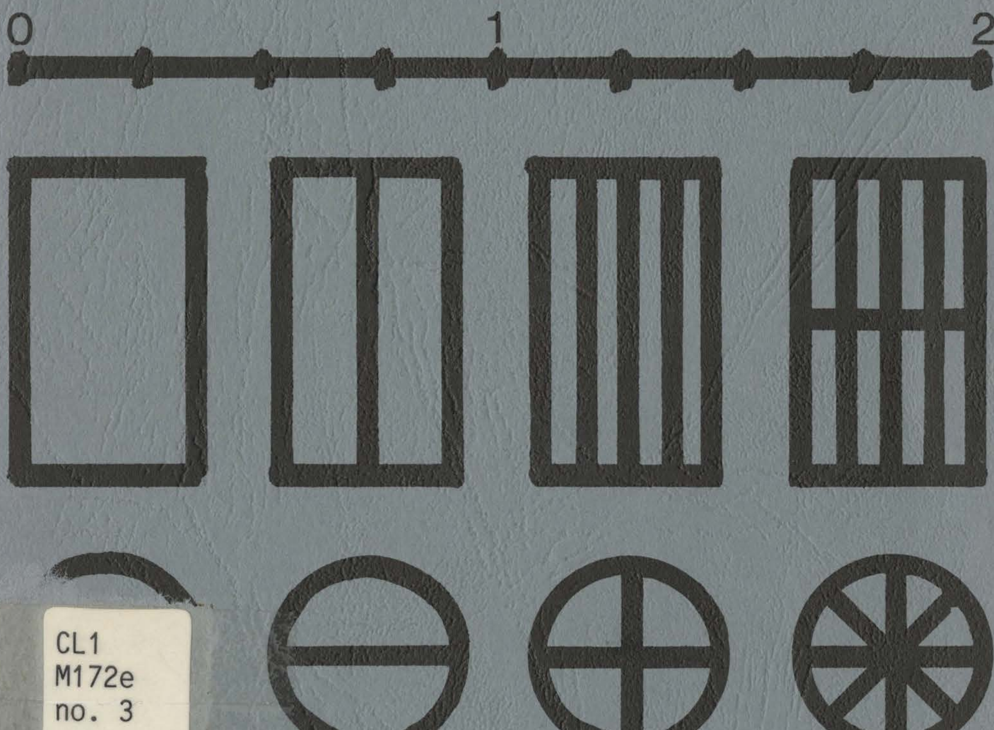

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MATHEMATICAL CONCEPTS, OPERATIONS, AND PROPERTIES RELATED TO THE SYSTEM OF RATIONAL NUMBERS

I. Introduction

Man, during past years, has invented and has structured systems of numbers to serve special needs. Mathematicians have developed mathematical properties or principles governing the structure of each system of numbers and the operations on the numbers.

The systems of natural numbers and positive whole numbers can be interpreted as the basic or foundational systems. The system of rational numbers and the system of integers have been structured to express new number ideas and to solve problems involving these different number ideas. As man finds it necessary to live in and to interact with a two-way society, the system of positive and negative integers was structured so that numbers were ideas of absolute value or count and of directional movement.

Pupils in today's contemporary world encounter daily some application of these four number systems. During the elementary school years, teachers are expected to provide instruction and resource experiences so that each pupil may discover and may develop understandings of the mathematical concepts (ideas), operations, and properties related to each of the four-number systems. As they progress in the elementary school, pupils are expected to advance from intuitive learning to the abstract level of mathematical reasoning and problem solving.

II. The Systems of Natural Numbers and Positive Whole Numbers

Since the systems of natural and positive whole numbers serve as the foundation upon which other systems of numbers have been invented and have been structured, it is important that teachers provide instruction and resourceful experiences so that each pupil may develop a conceptual interpretation and understanding of: (a) natural numbers, (b) positive whole numbers, (c) numerals as names for numbers, (d) the ordered sequence of natural and positive whole numbers, (e) the binary operations on these numbers and the inverse for each operation, and (f) the mathematical properties or principles governing operations on natural and whole numbers. Rational numbers have been called, at times, fractional numbers. It is recommended that pupils learn to use the term, rational numbers. The name for a rational number is called a fraction.

The System of Rational Numbers

As man encountered expansion in agriculture and trade, he found a new system of numbers that would enable him to think and to communicate number ideas related to congruent elements contained in sets. The early ideas and operations on rational numbers were associated with congruent sub-units of measure, such as, (a) a congruent part of a foot (1/2 ft., 3/4 ft.), (b) a measurement less than an acre, less than a mile, (c) a congruent part of a pound or a ton, (d) congruent parts of a unit of measure in cooking (1/2 cup, 1/2 tablespoon, 3/4 of a quart, or 1/3 of a dozen of eggs). In contemporary society, people are dependent upon an expanded structure and properties related to concepts and operations for rational numbers.

III. Intuitive Approach to Rational Numbers

Models such as rectangular pieces of cardboard, rectangular pieces of paper and a scaled number line can be used effectively as aids to pupils in discovering, intuitively, introductory concepts of rational numbers and their relation to concepts of whole numbers.

Examples

One-whole

1/2	1/2

1/2 is the name for the rational number. One-half of one whole.

$1/2 + 1/2 = 2/2 = 1$

A. 1. When a rectangular piece of paper is cut into two congruent parts (equal size parts), pupils can discover readily the meaning of the rational number named by the fraction 1/2 and the meaning of the rational number named by the fraction 2/2.

2. When reading the fraction 1/2, pupils should interpret and understand (a) that the natural number that is the denominator names the number of congruent

parts of one. And (b) that the natural number that is the numerator identifies the count of the number of congruent parts being considered.

3. It is important that pupils understand the mathematical concept to be applied to the denominator of a fraction and the mathematical concept to be applied to the numerator of a fraction.

B. The model $\boxed{1/4 \;\; 1/4 \;\; 1/4 \;\; 1/4} = 4/4 = 1$ is an illustration of aids which may be used to help pupils to discover the relation between one whole and four congruent parts of the whole.

1. When pupils read the fraction 3/4, they must think of three congruent parts of a whole changed to four congruent parts.
2. The rational number named by the fraction 4/4 is to be interpreted as a set of four one-fourths of one whole.
   (a) Pupils should discover and should understand the mathematical meaning of the statement $4/4 = 1$.

C. A scaled number line can be a useful aid for helping pupils to discover and to understand the mathematical concepts related to rational numbers and to the names for a rational number.

A		1/2		1B				2C
0	1/4	2/4	3/4	4/4	5/4	6/4	7/4	8/4

1. The line segment $\frac{A}{0}$ ——— $\frac{B}{1}$ is the base unit.
2. The meaning of the rational number named by the fraction 1/4 is: One congruent part of the base unit that has been converted into four congruent parts.
3. The meaning of the rational number named by the fraction 3/4 is: Three congruent parts of the base unit that have been converted into four congruent parts.
4. The meaning of the rational number named by the fraction 4/4 is: Four congruent parts of the base unit that have been converted into four congruent parts.
   a. 4/4 is the name for the concept: 4 one-fourths of one.
   b. 1 is the numeral for the number one: A whole base unit.
   c. Pupils should understand the mathematical concepts expressed by the statement: $4/4 = 1$
5. Teachers are expected to help pupils to discover and to understand the mathematical concepts related to equivalent fractions as different names for the same rational number.
   a. $1/2 = 1/4 + 1/4$ or $1/2 = 2/4$
   b. $4/4 = 1/4 + 1/4 + 1/4 + 1/4$ or $4/4 = 1$

D. The model for a scaled number line can be used to illustrate:

1. The cardinal count of one line segment $\frac{A}{0}$ ——— $\frac{B}{1}$ ——— $\frac{C}{2}$ and the cardinal count of two congruent line segments. $(\underset{AB}{\rule{2em}{0.4pt}} + \underset{BC}{\rule{2em}{0.4pt}})$
2. 1/2 is a fraction that names 1 line segment, when a line segment has been divided into two congruent parts.
3. 3/4 is a fraction that names the count of a subset of congruent parts when a base line segment has been partitioned into four congruent parts.
4. The fraction 4/4 and the numeral 1 are different names for the cardinal number, **one.**
5. The fraction 3/2 is the simplest or standard fraction for the count of 6 one-fourths of 1. $1/4+1/4+1/4+1/4+1/4+1/4= 1/2 +1/2 + 1/2 = 3/2$
6. Pupils can learn from their use of the intuitive process, these ideas:
   a. A rational number is a mathematical concept of one or more congruent units of one natural or whole number,

      2/2 4/2 6/2
      1 2 3

   b. The numeral that is the **denominator** names the number of congruent parts of one whole unit or set.
   c. The numeral that is the **numerator** names the count of the congruent parts being considered.

7. Pupils are expected to discover and to understand the mathematical reason why a fraction, such as, 5/8, may be expressed as an ordered pair of natural numerals, such as (5, 8):
    a. The first of a pair of numerals names the number of congruent parts of a given whole being considered.
    b. The second of a pair expresses the number of congruent parts of 1.

IV. Equivalent Fractions

A. Pupils are expected to discover and to understand that fractions are names for rational or fractional numbers.

B. Pupils are expected to discover and to understand that there are equivalent names or fractions for a rational number.
  1. $1/2 = 2/4 = 3/6 = 4/8 = 5/10 = 6/12 \ldots$
  2. The mathematical concept of the rational number may be expressed or be named by different fractions,
     $1/2 = 2/4$, is to be interpreted that two one-fourths is equivalent to one one-half of the same base unit.

C. Teachers are expected to help pupils to discover and to understand that equivalent fractions are names for the same rational number, however, when 3/4 is changed to the fraction 9/12, each 1/4 has been partitioned into 3 congruent parts. Each part is 1/12 of 1 and instead of 3 one-fourths, the new count is 9 one-twelfths of 1.

D. Pupils should be given the opportunity to learn the meaning of the **Rule of One** when converting a given fraction to one of its equivalents.
  $3/4 = 1 \times 3/4 = 3/3 \times 3/4 = 9/12$

E. Pupils should discover and should understand the mathematical concepts to be applied when changing a fraction to its equivalent, simplest form:
  1. $9/12 = 3/12 + 3/12 + 3/12 = 1/4 + 1/4 + 1/4 = 3/4$
  2. The simplest fraction is a fraction that names the count of the largest congruent parts of a given base unit that has the count of one.
     $3/12 = 1/4$
  3. A fraction that has a numerator and a number which are relative primes is called a standard fraction. (It is the simplest fraction.)

F. It is recommended that when pupils convert an improper fraction to a whole or mixed number that they rewrite the improper fraction so as to use the **Rule of One.**
  1. $16/4 = 4/4 + 4/4 + 4/4 + 4/4$
     $= 1 + 1 + 1 + 1 = 4$
  2. $18/16 = 4/4 + 4/4 + 4/4 + 4/4 + 2/4 = 1 + 1 + 1 + 1 + 2/4 = 4\text{-}2/4$
     (a) Pupils should be motivated to think, "How many subsets of 4/4 are contained in 16/4?"
     (b) Converting an improper fraction to a mixed numeral by dividing the numerator by the denominator is an abstract rule or conversion method.

G. It is important that teachers help pupils to discover and to understand the mathematical concepts to be applied when converting a mixed numeral to an improper fraction.
  1. $2\ 2/3 = 1 + 1 + 2/3 = 3/3 + 3/3 + 2/3 = 8/3$ For each one pupils should understand that 3/3 is another name for 1. $1 = 3/3$; $1 + 1 = 3/3 + 3/3 = 6/3$; $6/3 + 2/3 = 8/3$

H. A decimal fraction is a fraction where the denominator is expressed in decimal notation.
  1. .5 The numerator names 5 one-tenths of one.The denominator names the decimal value of each of the units named by the numeral ($.1 + .1 + .1 + .1 + .1 = .5$).
  2. $.25 = 2\ (.1) + 5\ (.01) = .2 + .05 = .25$

I. It is recommended that when pupils convert a common fraction (3/4) to the form of an equivalent decimal fraction, that they think, 1. There are how many 1/4's of 1 given? (3) 2. What decimal fraction is the equivalent name for 1/4? (.25)
  3. What is the decimal fraction that is the equivalent for 3/4?
     $1 = 1.00$; $1.00/4 = .25$
     $3/4 = 3(1.00/4) = 3(.25) = .75$

J. It is recommended that pupils discover and understand decimal fractions by using a scaled number line and partitions based on .1, .01, .001 etc.

1. Pupils may understand decimal fractions more quickly, when they extend decimal notation to the right of place-value one.

Ten	One	Tenths	Hundredths	Thousandths
10	1	.1	.01	.001

V. Binary Operations of a Pair of Positive Rational Numbers

A. Addition

1. Addition of a pair of rational numbers is a mathematical operation to find the standard or unique number and its fraction.
   a. Pupils should understand that addition is an operation on the numbers named by the numerators. The denominators identify the set of congruent parts.
   $$1/4 + 2/4 = \frac{1+2}{4} = 3/4$$
   b. When the fractions name unlike congruent parts the first step is to convert each fraction so that the pair of fractions name similar denominators.
   $$3/4 + 2/3 = (1 \cdot 3/4) + (1 \cdot 2/3) = (3/3 \times 3/4) + (4/4 \times 2/3)$$
   $$= 9/12 + 8/12 = 17/12 = 12/12 + 5/12 = 1 + 5/12$$
   $$= 1\text{-}5/12$$
   c. After pupils understand the mathematical concepts involved in (b) they should be encouraged to discover the mathematical concepts related to:
   $$a/b + c/d = ad/bd + bc/bd, \text{ if b and d} \neq \text{zero}$$
2. Pupils are expected to discover and to understand the mathematical properties that apply to addition on a pair of positive rational numbers.
   a. Identify property $0 + 3/4 = 3/4$; $3/4 + 0 = 3/4$
   b. Closure property $3/8 + 2/8 = 5/8$; $1/2 + 1/4 = 3/4$
   c. Commutative property $3/8 + 2/8 = 2/8 + 5/8$;
   $1/2 + 1/4 = 1/4 + 1/2$
   d. Associative property $3/4 + 1/2 + 5/8 = 5/8 + 3/4 + 1/2$
3. Teachers are expected to help pupils to understand the mathematical concepts related to addition when the binary operation is on a pair of mixed numerals.
   a. $12\,2/3 + 5\text{-}5/8 = \square$
   b. Pupils should learn to apply the grid algorism to find the replacement for $\square$.
   $$\begin{array}{r} 12\ 2/3 \\ +5\ \ 5/8 \\ \hline 1\ 7/24 \\ 17 \\ \hline 18\ 7/24 \end{array} \qquad \begin{array}{l} 2/3 + 5/8 = 16/24 + 15/24 = 31/24 \\ = 24/24 + 7/24 = 1\ 7/24 \\ \\ 12 + 5 = 17 \\ \\ 1\ 7/24 + 17 = 18\ \ 7/24 \end{array}$$
   c. Pupils should be encouraged to apply the horizontal or open sentence algorism.
   $$12\ \ 2/3 + 5\,5/8 = (12 + 5) + (2/3 + 5/8)$$
   $$= 17 + (16/24 + 15/24)$$
   $$= (17+1) + 7/24$$
   $$= 18\ \ 7/24$$
4. Pupils should be encouraged to use different algorisms to verify the accuracy of the addition operation.
5. After they understand addition as a binary operation on a pair of positive rational numbers, pupils should learn to use the formula
   $$a/b + c/d = ad/bd + bc/bd, \text{ if b and d} \neq 0.$$
   a. $7/8 + 3/4 = 28/32 + 24/32 = 52/32 = 32/32 + 20/32$
   $= 1 + 20/32 = 1\ \ 20/32$

b. If the mixed numeral is to be expressed in the simplest form, then the standard equivalent for
$20/32$ is $5/8$ $\quad 1 + 20/32 = 1 + 5/8 = 1\ 5/8$

B. Subtraction (Rational Numbers)

1. Teachers are expected to help pupils to discover and to understand the relation of subtraction on a pair of whole numbers to subtraction as a binary operation on a pair of positive rational numbers.
   a. $7 - 4 = 3$
   b. $7/8 - 4/8 = 3/8$
2. Teachers are expected to help pupils to understand the mathematical concepts when:
   a. Subtraction is an operation on a set or on two sets.
   b. Subtraction is an operation on a pair of positive rational numbers.
3. It is recommended that teachers help pupils to understand subtraction on a pair of rational numbers is an inverse operation of addition.
   a. $3/8 + 4/8 = 7/8$ $\quad$ 3 (one-eights) + 4 (one-eights) = 7 (one-eights)
   b. $7/8 - 3/8 = 4/8$ $\quad$ can be expressed as $7/8 = 3/8 + 4/8$
   c. $7/8 - 3/8 = \square$ $\quad$ can be expressed as $3/8 + \square/8 = 7/8$
4. Teachers are expected to help pupils to discover and to understand the reason why subtraction applies only to a pair of positive rational numbers named by fractions with the same denominator
   a. $7/8 - 1/2 = \square$; $\quad 7/8 - 4/8 = \square$; $\quad 4/8 + \square/8 = 7/8$
   b. $15/16 - 3/4 = \square$ $\quad 15/16 - 12/16 = \square$;
   $12/16 + \square/16 = 15/16$
5. After pupils have developed an understanding of subtraction as an operation on a pair of positive rational numbers, they should be motivated to discover the mathematical concepts related to the two formulas:
   a. $c/d - a/d = b/d$, if $d \neq$ zero and if $a/d + b/d = c/d$
   b. $c/d - a/b = cb/db - ad/db$ If d and $b \neq$ zero and $ad/db < cb$
6. Pupils should understand the mathematical concepts related to the two algorisms to be used to find a solution:
   a. Grid algorism $\begin{array}{r} 7/8 \\ -\underline{3/4} \\ \square \end{array}, \begin{array}{r} 7/8 \\ -\underline{6/8} \\ \square \end{array}, \begin{array}{r} 7/8 \\ -\underline{6/8} \\ 1/8 \end{array}, \begin{array}{r} 6/8 \\ +\underline{1/8} \\ 7/8 \end{array}$
   b. Open sentence or horizon algorism
   $7/8 - 3/4 = \square \quad 7/8 - 3/4 = 7/8 - 6/8 = 1/8$

   Additive inverse: $7/8 - 6/8 = 6/8 + 1/8 = 7/8$
7. Pupils are expected to discover and to understand the mathematical concepts to be applied when subtraction is a binary operation on a pair of mixed numerals.
   a. $\begin{array}{r} 20\ 1/8 \\ -\underline{16\ 5/8} \\ \square \end{array} = \begin{array}{r} 19\ 9/8 \\ -\underline{16\ 5/8} \\ \square \end{array}$ $\quad$ b. $\begin{array}{r} 21\ 1/4 \\ -\underline{8\ 3/4} \\ \square \end{array} = \begin{array}{r} 20\ 5/4 \\ -\underline{8\ 3/4} \\ 12\ 2/4 \end{array}$
   (b.1) $8\text{-}3/4 + 12\text{-}2/4 = 20\text{-}5/4 = 21\text{-}1/4$
8. Teachers are expected to help pupils to understand the mathematical reason why subtraction is the additive inverse operation on a pair of positive rational numbers.
   a. $11/16 - 5/16 = \square$ to $5/16 + \square/16 = 11/16$
   b. The additive operation is only on the numerators of a pair of fractions.
   (b.1) $7/8 - 3/8 = \square \quad 3/8 + \square/8 = 7/8$
   (b.2) $15/16 - 12/16 = \square$ to $15/16 - 12/16 = \square$
   $= 12/16 + \square/16 = 15/16$
   $12/16 + 3/16 = 15/16$
9. When subtraction is on a pair of mixed numerals, pupils should be discouraged to express each of the mixed numerals as an improper fraction, before applying subtraction.

10. Teachers are expected to help pupils to discover and to understand the reasons why the identity property, the commutative property, and the associative properties do not apply to subtraction as a binary operation on an ordered pair of positive rational numbers.

VI. Mathematical Concepts and Properties Related to Multiplication and Its Inverse Operation, Division, On An Ordered Pair of Positive Rational Numbers

A. Multiplication (An operation of rapid addition)

1. Pupils are expected to understand that multiplication is a mathematical operation on an ordered pair of positive rational numbers to find the unique or standard equivalent number.
2. Teachers are expected to help pupils to understand multiplication as a mathematical and a mental operation. The operation is governed by established mathematical properties.
3. It is recommended that concrete objects and models be used during the initial stage of conceptualizing by intuition the foundational concepts related to multiplication as an operation on a pair of rational numbers.
4. Pupils after reading the open sentence, $4 \times 3/8 = \square$, should be encouraged to think:
   a. What question is asked by, $4 \times 3/4 = \square$?
   b. What does 4 identify?
   c. What does 3/8 identify?
   d. What does the symbol, $\square$, stand for?
   e. What is the meaning of the sentence?
   $4 \times 3/8 = 3/8 + 3/8 + 3/8 + 3/8 = 12/8$
   $= 8/8 + 4/8 = 14/8 = 11/2$
   f. What is the meaning of the sentence?
   $4 \times 3/8 = 4 \times 3/8 = 12/8 = 8/8 + 4/8$
   $= 1\ 4/8 = 1\ 1//2$
5. Pupils after reading the open sentence, $4 \times 3\ 3/8 = \square$ should learn to think
   a. What question is asked?
   b. What does the 4 identify?
   c. What does 3 3/8 represent?
   d. What does the symbol, $\square$, stand for?
   e. What mathematical concept is to be applied to the product?
   f. What mathematical change is to be made in mathematical question when $4 \times 3\ 3/8 = \square$ is changed to $3\ 3/8 \times 4 = \square$
   g. What concepts are to be applied to:
   $(4 \times 3) + (4 \times 3/8) = \square$
6. Pupils when using the grid algorism for $4 \times 3\ 3/8 = \square$ should conceptualize,

3 3/8	
×4	$4 \times 3/8 = 12/8 = 8/8 + 4/8 =$
1 4/8	$1 + 4/8 = 1\ 4/8$
12	$4 \times 3 = 12$
13 4/8	

B. Multiplication (An operation of partitioning)

1. Pupils, after reading the open sentence $1/2 \times 3/4 = \square$ should learn to think:
   a. What question is asked?
   b. How many congruent parts are given?
   c. What is the name of each congruent part?
   d. Why is the symbol, $\times$, to be interpreted as of?
   e. What concept is to be applied to 1/2?
   f. What question is asked by $1/2 \times 3/4 = \square$?
   g. What meaning is to be applied to, $3\ (1/2 \times 1/4) = 3\ (1/8) = 3/8$?
2. Pupils may interpret with greater meaning the multiplication operation on a pair of rational numbers by the algorisms.
   a. $1/2 \times 3/4 = 3(1/2 \times 1/4) = 3/8$
   b. $2/3 \times 5/8 = 5(2/3 \times 1/8) = 10/24$

3. Pupils need concrete models and professional instruction when conceptualizing the abstract formula,
   $a/b \times c/d = ac/bd$, if b and d $\neq$ zero
4. It is important that pupils read and interpret the open sentence $9\ 5/8 \times 24\ 2/3 = \square$ before they apply multiplication to find the correct solution.
   a. There are 9 addends with the number 24 2/3 and 5/8 of another addend with the number 24 2/3.
   b. There are two mathematical concepts involved, partitioning, $(5/8 \times 24\ 2/3)$, and rapid addition, $(9 \times 24\ 2/3)$.
   c. Frequently pupils should be asked to write a mathematical sentence that describes the concept used for each step when performing the operation

$$\begin{array}{rl} 24\ 2/3 & \\ \underline{9\ 5/8} & \\ 10/24 & \quad 5/8 \times 2/3 = 10/24 \\ 15 & \quad 5/8 \times 24 = 15 \\ 6 & \quad 9 \times 2/3 = 6 \\ \underline{216} & \quad 9 \times 24 = 216 \\ 237\ 10/24 & \end{array}$$

5. Cancellation when used in the operation to find the product of a pair of rational numbers is an abstract operation. It is suggested that the application of cancellation be postponed until pupils have studied factors and factoring.
   a. Non-cancellation algorism: $3/8 \times 16/24 = 48/192 = 1/4$
   b. Cancellation algorism: $3/8 \times 16/24 = \dfrac{\cancel{3}}{\cancel{2} \times \cancel{2} \times \cancel{2}} \times \dfrac{\cancel{2} \times \cancel{2} \times \cancel{2} \times \cancel{2}}{\cancel{2} \times 2 \times 2 \times \cancel{3}}$

$$= \frac{1}{2 \times 2} = 1/4$$

6. Teachers are expected to help pupils to discover and to understand the mathematical properties for multiplication on rational numbers.
   a. Identify property. $1 \times 3/4 = 3/4$
   b. Closure property. $2/3 \times 5/8 = 10/24$
   c. Commutative property. $3/4 \times 5/6 = 5/6 \times 3/4$
   d. Associative property. $2/3 \times 1/2 \times 7/8 = 1/2 \times 2/3 \times 7/8$
   e. Distributive property over addition.
      $2/3 \times (5/8 + 3/4) = (2/3 \times 5/8) + (2/3 \times 3/4)$
   f. Reciprocal property. $5/6 \times 6/5 = 1$
7. Teachers are expected to help pupils to understand the relation between the multiplication properties for whole numbers and the multiplication property for rational numbers.

C. **Division** (A Binary Operation on an Ordered Pair of Rational Numbers)
1. Teachers are expected to help pupils to discover and to understand that division is an inverse operation of multiplication.
   a. $6 \times 2/3 = 12/3$ $\quad 12/3 \div 2/3 = 6$
      12/3 may be partitioned into 6 subsets of 2/3 each.
   b. $21/8 \div 7/8 = \square$ $\quad \square \times 7/8 = 21/8$ $\quad 3 \times 7/8 = 21/8$
2. Since division is an inverse of multiplication, the operation may be understood as repeated subtraction.
   (a) $21/8 \div 7/8 = [(21/8 - 7/8) - 7/8] - 7/8$
   When the repeated subtraction algorism is used, the pair of rational numbers must have like denominators
3. When division is an operation on an ordered pair of mixed numerals, each numeral must be named by a fraction.
   $8\ 1/2 \div 2\ 3/4 = \square$ $\quad 34/4 \div 11/4 = 3\ 1/11$

4. Pupils should discover and should understand that when the dividend or product is a number that is larger than the known factor or divisor, then the quotient

will be one or greater than one.

(a) $6/8 \div 2/8 = 3$ (b) $15/16 \div 7/16 = 2\text{-}1/7$

(b) Teachers are expected to help pupils to understand the mathematical reason for this property or principle.

5. Pupils should discover and should understand that when the dividend or product is a number that is less than the divisor, then the quotient will be less than one.

(a) $1/2 \div 3/4 = \square$ $1/2 \div 3/4 = 2/4 \div 3/4 = 2/3$

1/2 does not contain 3/4

$2/3$ of $3/4 = 2/4$ 1/2 contains 2/3 of 3/4 or 2/4

(b) Teachers are expected to help pupils to understand the mathematical reason for this property or principle.

6. Pupils should be encouraged to experiment to discover why there is a **closure property** for division as a mathematical operation on an ordered pair of rational numbers.

7. After pupils have developed an understanding of division as applied to an ordered pair of rational numbers they should be expected to discover and to understand the reciprocal algorism.

(a) $8/4 \div 2/4 = \square$ $8/4 \times 4/2 = 32/8 = 4$

(b) $7/12 \div 3/12 =$ $7/12 \times 12/3 = 7/3 = 2\ 1/3$

$7/12 \div 3/12 = 2\ 1/3$

$(2\ 1/3 \times 3/12 = 7/3 \times 3/12 = 7/12$

(c) Pupils should understand that dividing by a rational number (not zero) produces the same result as multiplying the dividend by the reciprocal for the mul tiplier or known factor.

8. Pupils should be motivated to verify the accuracy of the solution for a division problem by multiplying the known factor or divisor by the obtained quotient to determine if the product is equivalent to the known dividend.

(a) $11/2 \div 3/4 =$ $11/2 \div 3/4 = 22/4 \div 3/4 = 22/3 = 7\text{-}1/3$

(b) Verifying accuracy of the solution

$7\text{-}1/3 \times 3/4 = 22/3 \times 3/4 = 66/12 = 11/2$

7-1/3 is the correct solution.

D. Mathematical Concepts and Operations Related to Rational Numbers Named by Decimal Fractions.

1. Pupils have learned, from their study of rational numbers that the numbers are mathematical and mental concepts. Fractions are names for rational numbers.

(a) 7/8 is a fraction or a fractional numeral.

(b) (7, 8) is an ordered pair of natural numbers which is another form for naming a rational number.

2. Decimal fractions or decimals are also used as names for positive rational numbers. (.5), (.25), (.875)

3. Simon Stevin, a Belgian surveyor, made a valuable contribution to mathematics by structuring an extension of the decimal numeration system to the right of the ones position to name rational numbers with a denominator of a tenth of one or of a power of a tenth of one.

(a)

Ten	One	Tenth	Hundredth	Thousandth
10	1	1/10. of 1	1/100 of 1	1/1000 of 1
10.	1.	.1	.01	.001

(b) A symbol generally used to identify the separation of one from a tenth or tenths is a dot.

.5 = 5 tenths

.25 = 2 tenths + 5 hundredths

.45 = 4 tenths + 5 hundredths

(c) A strip of graph paper containing ten equivalent squares and subsets of ten squares is a useful model in helping pupils to discover and to understand decimal fractions as names for positive rational numbers with a value less than one.

(d) The form, decimals, is used more frequently rather than decimal fractions.

(e) A mixed numeral, 3.85 is also called a decimal.
(3 ones $+$ 8 tenths $+$ 5 hundredths)

(f) A scaled number line is another model that may be used to help pupils discover mathematical concepts related to positive rational numbers named in the form of decimal fractions.

4. Pupils should experience no difficulty in understanding how the decimal grid for addition of whole numbers can be used, when place-value positions are structured to the right of the ones position, when addition involves decimal fractions and mixed numerals.

(a) $.46 + .27 + .18 =$

$.46 + .27 + .18 = .91$

	one	tenths	hundredths
		4	6
		2	7
+		1	8
		9	1

(b) $27.64 + 69.87 =$

$27.65 + 69.87 = 97.52$

ten	one	tenths	hundredths
2	7	6	5
6	9	8	7
9	7	5	2

(c) Pupils should understand, readily, that they use the standard addition combinations structured for whole numbers, and the decimal value assigned to the standard digits depends upon the place-value positions in which they are placed.

5. Pupils after understanding the application of the decimal grid algorism should be motivated to learn to apply the horizontal algorism.

(a) $.46+.27+.18 = .4+.2+.1+.06+.07+.08 = .7+.21 = .91$

(b) $27.65+69.87 = 2(10)+7(1)+6(.1)+5(.01)+6(10)+9(1) +8(.1)+7(.01)$
$= 2(10)+6(10)+7(1)+9(1)+6(.1)+8(.1)+5(.01) +7(.01)$
$= 80. + 16. + 1.4 + .12 = 97.52$

6. Teachers are expected to help pupils to discover and to understand that the basic properties for addition of whole numbers apply to rational numbers named by decimal fractions.

(a) Identity property $.6 + 0 = .6$, $0 + .8 = .8$
(b) Closure property $.5 + .4 = .9$, $.07 + .09 = .16$
(c) Commutative property $.7 + .2 = .2 + .7$
(d) Associative property $.8 + .6 + .4 = .4 + .8 + .6$

E. Subtraction

1. Pupils should experience little difficulty in understanding the relation of subtraction as a binary operation on a pair of natural numbers to subtraction as a binary operation on a pair of rational numbers named by decimal, fractions.

(a) $8. - 3. = \square$ (b) $.8 - .3 = \square$ (c) $.08 - .03 = \square$

8 in each sentence represents the sum and 3 in each sentence represents the given addend. $\square$ represents the unknown addend.

2. The decimal grid algorism will help pupils to understand how the mathematical concepts related to subtraction of whole numbers may be applied when subtraction is a binary operation on a pair of numbers named by decimal fractions and mixed decimals.

(a)

	o	t
	.7	6
−	.3	9
	.3	7

$.39 + .37 = .76$

(b)

	o	t	h
	3	.7	6
−	1	.8	9
	1	.8	7

$1.89 + 1.87 = 3.76$

3. Pupils should be motivated to apply the additive method when finding a solution for subtraction on a pair of decimal fractions or mixed decimal numerals.

(a) $.76 - .48 = \square$ $.48 + \square = .76$ $.48 + .28 = .76$
(b) $9.05 - 6.50 = \square$ $6.50 + 2.55 = 9.05$ $6.8 + 50 + 2.55 = 9.05$

F. Multiplication

1. Pupils should experience little difficulty in understanding multiplication when the first factor is a natural number and the second factor is a rational number

named by a decimal fraction.

(a) This operation should be interpreted as a short-cut operation for rapid addition of a given number of like factors.

$4. \times .2 = \square \quad .2 + .2 + .2 + .2 = \square \quad 4. \times .2 = .8 \, \square$

2. Pupils should experience little difficulty when multiplying a decimal fraction by a natural number, if they understand:

(a) The application of the standard combinations for multiplication as they are used in the place-value positions.

(b) The natural number names the number of rational numbers named by decimal fractions.

$5. \times .7 = \square \qquad 6. \times .07 = \square$

(c) That there will be as many decimal place-value positions in the product as are in the second factor, unless ones must be substituted for ten tenths.

$$\begin{array}{r} .34 \\ \times \underline{2.} \\ .68 \end{array} \qquad \begin{array}{r} .78 \\ \times \underline{2.} \\ 1.56 \end{array} \qquad \begin{array}{r} .645 \\ \times \underline{2.} \\ 1.290 \end{array}$$

.34 is a second factor
.78 is a second factor
.645 is a second factor

3. Multiplication as a binary operation on an ordered pair of rational numbers named by decimal fractions is somewhat more difficult to conceptualize.

(a) The operation involves partitioning rather than rapid addition.

(a.1)

	t	h
	.4	
	.3	
×	.1	2

The question to be asked is, "What is 3 tenths of 4 tenths?" Pupils should think, "3 tenths of 1 tenth is 3 hundredths, therefore, $.3 \times .4 = 4(.3 \times .1) = 4(.03) = .12$

(a.2)

0	t	h	th	tth
	.7	6		
×	.5	4		
			2	4
		2	8	
—	0	3	0	
	3	5		
	4	1	0	4

$.54 \times .76 = \square$

$.04$ of $.06 = 6(.04$ of $.01) = 6(.0004) = .0024$
$.04$ of $.7 = 7(.04$ of $.1) = 7(.004) = .028$
$.5$ of $.06 = 6(.5$ of $.01) = 6(.005) = .030$
$.5$ of $.7 = 7(.5$ of $.1) = 7(.05) = .35$

Pupils from their work with the decimal grid algorism may soon discover and understand:

That there will be as many place-value positions in the product with a value less than one as the sum of the less than one decimal positions in the two factors.

That they multiply as with whole numbers, and then determine the number of decimal position with a value less than 1 that is to be in the product.

That they may determine the smallest decimal place-value position to be in the product by finding the decimal value of the pair of standard digits in the smaller place-value in each of the factors.

(b) Pupils, from their understanding of multiplication as applied to a pair of whole numbers or to a pair of rational numbers named by decimal fractions, should experience little difficulty in understanding the mathematical concepts to be applied when finding the product for a pair of mixed decimal numerals. $4.86 \times 7.23 = \square$. (Multiplication is performed as if the numbers were whole numbers. The number of decimal place-value positions for the product is equal to the sum of the decimal place-value positions in the two factors. $4.86 \times 7.23 = 35.1378$)

(c) Teachers are expected to help pupils to discover and to understand how the mathematical properties for whole numbers apply to an ordered pair of

decimal fractions or mixed decimal numerals.

G. **Division** (rational numbers named by decimal fractions or mixed numerals)

1. Teachers are expected to help pupils to understand the reasons why many mathematical concepts related to division as a binary operation on an ordered pair of whole numbers may be applied when division is a binary operation on an ordered pair of decimal fractions or mixed numerals.
2. Teachers are expected to help pupils to learn to read and to interpret properly each open sentence or equation involving division before computation is used to find the appropriate solution.
3. Teachers are expected to help pupils to understand the mathematical rules or principles which may be applied to division involving an ordered pair of decimal fractions or mixed numerals to determine the placement of the decimal point in quotient.
4. Models of graph paper or of a scaled number line can be effective means by which pupils may discover the justification for operational rules related to division as an operation on an ordered pair of decimal fractions or mixed numerals.
5. Teachers are expected to help pupils to understand the mathematical concepts and operations related to each of the following types of open sentences requiring division to find a correct solution:

   (a) Division involving an ordered pair of decimal fractions and the quotient is a natural number.

   $.9 \div .3 = \square \qquad .9 \div .3 = 3.$

   $.75 \div .15 = \square \qquad .75 \div .15 = 5.$

   After reading the open sentence $.9 \div .3 = \square$ pupils should think:

   How many .3 are contained in .9?
   What number times $.3 = .9$?
   Why is the quotient a natural number?
   Where is the decimal point placed in the quotient?

   After reading the open sentence $.75 \div .15 = \square$ pupils should think:

   How many .15 are equal to .75?
   What number times the divisor equals the dividend?
   Why is the quotient a natural number?
   Where is the decimal point placed in the quotient?

   (b) Division involving an ordered pair of decimal fractions and the quotient is a number named by a mixed decimal numeral.

   $$.54 \div .12 = \square \qquad .12\overline{)\,.54}^{\;N} \qquad \begin{array}{r} 4.5 \\ .12\overline{)\,.54} \\ \underline{.48} \\ .060 \\ \underline{.060} \end{array}$$

   After reading the open sentence pupils should think:

   How many .12 are contained in .54?
   Is the division an even or uneven operation?
   Why will the quotient be a mixed numeral?
   What does 4. in the quotient express?
   What does .5 in the quotient express?

   (c) Division involving an ordered pair of decimal fractions and the quotient will be a decimal fraction.

   $$.4 \div .8 = \square \qquad \begin{array}{r} N \\ .8\overline{)\,.4} \end{array} \qquad \begin{array}{r} .5 \\ .8\overline{)\,.4} \end{array}$$

   After reading the open sentence $.4 \div .8 = \square$ pupils should think:

   How many .8 are contained in .4?
   What decimal part of .8 equals .4?
   Why is the quotient named by a decimal fraction?

6. Pupils may interpret more readily the mathematical concepts expressed by the quotient for an ordered pair of decimal fractions or mixed numerals if they first

express the divisor and the dividend by decimals having equivalent place value positions.

$.8 \div .04 =$ to $.80 \div .04 =$ $.04\overline{).8}$ to $.04\overline{).80}$

How many .04 are contained in .80?

Why will the quotient be named by a natural number?

$26. \div 3.25 =$ $3.25\overline{)26.}$ to $3.25\overline{)26.00}$

How many 325 hundredths are contained in 2600 hundredths?

Why will the quotient be named by a natural number? $26.00 \div 3.25 = 8.$

7. Teachers are expected to help pupils to understand the mathematical rule for determining where the decimal point is to be placed in the quotient when division involves a pair of decimal fractions or a pair of mixed numerals.
   (a) The number of decimal places in the quotient can be determined by subtracting the number of decimal places in the divisor from the number of decimal places in the dividend.

   $3.5625 \div .375 = 9.5$

   (b) When multiplying a pair of decimal fractions or mixed numerals, the number of decimal places in the product is equal to the sum of the decimal places in each factor.

   $9.5 \times .375 = 3.5625$
8. After pupils have developed an understanding of the mathematical concepts related to division as an operation on an ordered pair of decimal fractions or mixed decimal numerals, teachers are expected to help pupils to understand the algorithm by which the divisor is named by a natural number by multiplying the divisor and the dividend by a power of ten needed to change the divisor to a natural number.
   (a) $7.2 \div .8 = \square$

   $(7.2 \times 10) \div (.8 \times 10) = \square$ $8.\overline{)72.}$

   (The ratio relation between the dividend and the divisor remains unchanged.)

   (b) Teachers are expected to help pupils to understand how a caret ( ) may be used as an aid for determining the placement of the decimal point in the quotient.

   (The caret in the dividend points to the place in the quotient where the decimal point is to be placed.)

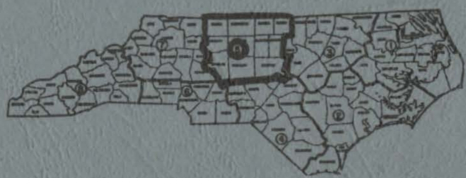
   $$8.25\overline{)825.} \text{ to } 825\overline{)825\,00.}^{\uparrow} \quad (\text{quotient } 1\,00.)$$

Basic Terms and Operations Related to Rational Numbers

It is important that teachers in each elementary school reach consensus concerning the precise mathematical understanding pupils should be expected to learn and to apply for each term or operation listed below.

1. Addition of rational numbers
2. Algorism
3. Conversion of a fraction to an equivalent decimal fraction
4. Conversion of a decimal fraction to an equivalent common fraction
5. Conversion of a natural number to an equivalent rational number
6. Conversion of a rational number to a natural number
7. Common fraction
8. Congruent parts
9. Decimal fraction
10. Denominator of a common fraction
11. Denominator of a decimal fraction
12. Division of rational numbers
13. Divisor
14. Dividend
15. Equivalent fraction for a rational number
16. Improper fraction
17. Mathematical operation on a pair of rational numbers

18. Mixed numeral
19. Multiplication of rational numbers
20. Natural number
21. Numerator of a common fraction
22. Numerator of a decimal fraction
23. Open number sentence
24. Ordered arrangement of rational numbers
25. Operation on an ordered pair of rational numbers
26. Product
27. Proper fraction
28. Properties for addition of rational numbers
    a. Identity property
    b. Closure property
    c. Commutative property
    d. Associative property
29. Properties for division of rational numbers
    a. Closure property
    b. Multiplicative reciprocal property.
30. Properties for multiplication of rational numbers
    a. Identity property
    b. Closure property
    c. Commutative property
    d. Associative property
    e. Distributive over addition property
    f. Distributive over subtraction property
    g. Reciprocal property
31. Quotient
32. Rational number
33. Similar fractions
34. Simplest fractional name for a rational number
35. Subtraction of rational numbers
36. Unit fraction



THE PIEDMONT ASSOCIATION

FOR

SCHOOL STUDIES AND SERVICES

GUIDE NUMBER FOUR

elementary mathematics

measures and measurement

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for

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MEASURES AND MEASUREMENT

INTRODUCTION

Units that are designed to help pupils to develop mental concepts of measures and the application of standard units of measure are essential in a balanced curriculum for elementary mathematics. The study of measures and measurement should enable pupils to discover and to understand that man's progress in various fields such as building, industry, scientific research, commerce, and government has been dependent upon improvements of units of measure and processes to determine with greater precision the measurement of physical quantities and operations on physical quantities. An appropriate and a functional understanding of measures and measurement is essential for intelligent citizenship and productive work in a rapidly advancing world characterized by nuclear fission, technology, and automation.

Pupils interest in and reason for pursuing their study of measures and measurement can be motivated by opportunities to examine and to explore their conceptual response to such questions as the following:

1. How do pupils, in each day of living, experience the application of measures and also their need for precision of measurement?
2. What confusion would result in a day's activities in a home, or in a school, or in a community, if there were no standard units of measure?
3. Why has modern man established laws and governmental agencies to regulate the consumer application of measures?
4. Why has man found the need to define standard units of measure and to construct measuring machines for determining very large and very small measurements?
5. How does an understanding of numbers and binary operations on numbers serve in understanding the relation between measurement and mathematics?
6. How is consumer economics related to measures and measurement?
7. In what way is the metric system of weights and measures simpler in structure and in application than the English — U. S. A. system of weights and measures?
8. What are some of the advantages to be obtained if all nations of the world legally adopt the metric system and a decimal monetary system?

PROFESSIONAL OPPORTUNITY OF TEACHERS

Teachers in each elementary school are expected to construct units on measures and measurement that will motivate and will assist each learner:

1. To develop a conceptual understanding of the difference and the relation between numbers and measurement.
2. To develop skill in making an estimate or a reasonable guess of the measure of objects and physical situations frequently encountered at home, at school, and in the community.
3. To determine the necessary steps to be taken to obtain a reliable measurement.
4. To understand the advantage of the metric system of weights and measures when applied locally, nationally, and internationally.
5. To acquire an understanding of the role of the Congress of the United States in defining standard units of weights and measures and the legal responsibility of each state to establish laws and agencies to govern the application of standard units of measure.
6. To acquire an understanding of services rendered to the states and to American citizens by the United States Bureau of Standards.

MEASURES AND MEASUREMENT IN THE PRIMARY GRADES

Pupils in each primary grade should be given frequent opportunity to observe, to examine, and to apply such English—U.S.A. units of measure as: yard, foot, inch, ounce, pound, quart, gallon, pint, cup, square inch, square foot, cubic inch, cubic foot. They should discover by observing and examining the structured scale of the Fahrenheit Thermometer for measuring temperature. Pupils should be encouraged to make an estimate of the measure of frequently experienced objects and then apply a unit of measure to verify the accuracy of their estimated measurement.

It is important that teachers provide pupils with units of measure in the metric system in order for them to conceptualize their measure and to compare the units of length, capacity, and weight with similar units in the English — U. S. A. system of weights and measures. Such experience is more productive in conceptual understanding than looking at pictures of units of measure or reading tables of weights and measures.

Pupils will begin to interpret with fuller meaning the functions of indirect measurement when they are given the opportunity to observe, to examine, and to use such measuring instruments as a weighing scale, a thermometer, a watch or a clock. Interpreting weight as the measure of the pull of the earth's gravity on an object becomes more meaningful when pupils lift objects, guess their weight, and then find a weight by using a set of scales. Reading the hands of a clock becomes more meaningful as pupils grasp the concept that time is movement and only in a forward direction. A measurement of time expresses a comparison between a clock or watch and the relation between the earth and the sun.

It is recommended that each classroom be equipped with the following instruments: a meter stick, a yardstick, a foot ruler, a liter, a gallon, a pint, a Fahrenheit thermometer, a Centigrade thermometer, a gram, a kilogram, a pound, an ounce, a square inch, a square foot, a cubic inch, a cubic foot, a cubic decimeter, a cubic centimeter, a 12-hour watch, or a 24-hour clock, and a calendar. These instruments are more effective in developing concepts of measures and measurement than pictures and tables. Frequent trips to different stores will provide experiences which will contribute to improvement in understanding the meaning and function of measures and measurement in a community.

MEASURES AND MEASUREMENT IN THE INTERMEDIATE GRADES

It is important that the mathematics curriculum offer units and experiences that will assist pupils to improve their understanding of the difference between number and measurement. **A number is the cardinal property of a set of elements. A measurement expresses the determined ratio between a given magnitude or continuous quantity and an obtained measurement.** Pupils need experience by which to discover that mathematical operations such as addition and subtraction are on **numbers** and not on measurements.

Pupils should be provided instruments and objects for discovering and properly interpreting the difference between a direct measurement and an indirect measurement. From their findings pupils should discover why certain formulas to obtain an indirect measurement may be more practicable than attempting to apply directly a required unit of measure. The measurement of the area of a surface may be obtained more quickly by applying the formula, $A = hb$. (For each unit of the linear measure of the base a square unit is substituted and the measure of the height indicates the number of rows of square units applied to the linear measurement of the base.) It is important that pupils be given instruments and objects which will help them to discover and to understand the meaning and importance of precision of a measurement. Pupils should be motivated to do library research or field research to discover the function of standard units of measure and measuring instruments in scientific research, in industry, in business, in the professions, in government, and in public education.

On completion of the last year in an elementary school each pupil should be expected to have acquired understanding, such as: (a) the structure and ordered relations of the units in the metric system of measures (b) the structured scale of the Centigrade thermometer, (c) the scale and merit of the 24-hour watch or clock (d) the reading obtained from a gas meter, a water meter, and an electric meter.

Pupils should be motivated to engage in research to obtain information related to: (a) the merits of an internationally legalized decimal monetary system (b) the application of standard units of measure for exploration in outer space and oceanic space (c) the importance of the use of electronic computers in industry, in business, and in state and federal government and (d) the application of measures and measurement to qualitative magnitudes.

MEASUREMENT, NUMBERS, AND MEASURE

It is important that teachers provide learning situations and professional leadership that will assist pupils in developing a reliable understanding of each of the following terms and their application: (a) measurement (b) numbers (c) measure (d) direct measurement, and (e) indirect measurement.

Measurement:

A measurement identifies the determined ratio between a given object or magnitude and a selected standard unit of measure. A measurement is an approximation. The smaller the unit of measure to be applied, the nearer the measurement approaches an absolute value. A measurement is the name for a measure together with the applied standard unit.

If pupils wanted to find the measure of length of one side of a classroom, they would undertake the following operational steps: (a) select a unit of measure (b) apply the unit to the measure and (d) express the result by the appropriate measurement.

If the measurement was 14 feet, 6 inches, the numbers 14 and 6 identify the cardinal count of the applied units and the words feet and inches identify the units of measure. Pupils should be asked to prepare a list of samples of direct measurement that may be obtained in the classroom, in the school, in the home, and in the community.

Measure:

A measure is a mental concept. It is obtained by applying a selected unit of measure to a given object or magnitude.

Numbers:

Numbers are mathematical concepts. They exist only in a person's mind. A whole number identifies the cardinal property (the count) of a given set of whole elements. A rational number is also a mathematical concept. It identifies the relation of the count of a subset of congruent parts to the count of a given set of congruent parts. (The fraction 3/4: 3 is the name for the number of congruent parts in the subset, and 4 names the number of congruent parts in the given set.)

SYSTEMS OF WEIGHTS AND MEASURES

ENGLISH SYSTEM · U. S. A.

The oldest system of weights and measures is called the English System. Since this system is the officially adopted system in the United States of America it is sometimes called the English — U. S. system. The Parliament of England enacted into law in 1824 the system of weights and measures recommended by the Royal Commission. The adopted standard units are:

The Imperial Yard — 36 inches
The Imperial Gallon — 217.27 Cubic Inches
(The U. S. Standard Gallon equals 231 Cubic Inches)
The Imperial Bushel — 2218.19 Cubic Inches
(The U. S. Bushel equals 2150.42 Cubic Inches)

The Imperial Avoirdupois Pound — 16 Ounces
(This is the standard unit of weight most commonly used in Great Britain and the United States.)

The standard units and related units most frequently used are given in the following tables:

Linear Measure

12 inches (")	=	1 foot (ft.) (')
3 feet	=	1 yard (yd.)
5-1/2 feet	=	1 rod (rd.)
660 feet	=	1 furlong (fur.)
5280 feet	=	1 statute mile (mi.)

Square Measure

144 square inches	=	1 square foot (sq. ft.)
9 square feet	=	1 square yard (sq. yd.)
30-1/4 square yards	=	1 square rod (sq. rd.)
160 square rods	=	1 acre (A.)
640 square acres	=	1 square mile (sq. mi.)

Cubic Measure

1728 cubic inches	=	1 cubic foot (cu. ft.)
27 cubic feet	=	1 cubic yard (cu. yd.)
144 cubic inches	=	1 board foot (bd. ft.)

Avoirdupois Weight

437.5 grains	=	1 ounce (oz. avoir.)
16 ounces	=	1 pound (lb. avoir.)
100 pounds	=	1 short hundred weight (sh. cwt.)
2000 pounds	=	1 short ton (sh. tn.)
2240 pounds	=	1 long ton (l. tn.)

Liquid Measure

4 gills	=	1 pint (pt.)
2 pints	=	1 quart (qt.)
(U.S. quart	=	57.75 cu. in.)
4 quarts	=	1 gallon (1 gal.)
(U.S. gallon	=	231 cu. in.)

Nautical Measure (used on the sea)

6 feet	=	1 fathom
6080.27 feet	=	1 nautical mile
1 nautical mile per Hour	=	1 knot (speed)

THE METRIC SYSTEM

The metric system is a logical and scientific system. This system of weights and measures was invented and was structured by a group of French scientists appointed by the government of France. The metric system was adopted by the French Government in 1799. It is the legally adopted system of most countries of the world, except Great Britain, Canada, and the United States of America. It is legal to use the metric system in the United States and it is used widely in scientific research, industry, and government agencies.

The base unit of measure in the metric system is the **Meter.** The standard base units related to the meter are:

1. The Liter (unit of capacity)
2. The Gram (unit of weight).

The volume of the liter equals one cubic decimeter (dm^3). The gram equals the weight of one cubic centimeter of water at 4 degrees Centigrade. The new standard for the meter is based on the wave length of the orange light given off by Krypton 86.

The specific advantage of the metric system is that it is a decimal system. Each unit of measure has a decimal relation to the next larger unit or to the next smaller unit. Greek prefixes have been selected to identify multiples by ten of each base unit. Latin prefixes are used to name decimal divisions of each standard base unit. (Refer to the chart, page 4.)

THE METRIC SYSTEM

1000	100	10	1	.1	.01	.001
kilometer	hectometer	decameter	meter	decimeter	centimeter	millimeter
kiloliter	hectoliter	decaliter	liter	deciliter	centiliter	milliliter
kilogram	hectogram	decagram	gram	decigram	decigram	milligram

(mega = 1,000,000 myria = 10,000 micro = 0.000001)

Relation between Metric System and Commonly Used U. S. Units

Length	**Metric Units**	**U. S. Units**
1 centimeter	= 0.3937 inch	1 inch = 2.54 centimeters
1 decimeter	= 3.94 inches	1 foot = 3.048 decimeters
1 meter	= 39.37 inches or 3.281 feet	1 foot = 0.3048 meter
1 meter	= 1.094 yards	1 yard = 0.9144 meter
1 kilometer	= 0.621 statute mile	1 statute mile = 1.609 kilometers
1 kilometer	= 0.540 nautical mile.	1 nautical mile = 1.853 kilometers
Weight		
1 gram	= 0.035 avoir. ounce	1 avoir. ounce = 28.35 grams
1 kilogram	= 2.2046 avoir. pounds	1 avoir. pound = .4536 kilogram
1 metric ton	= 1.102 short ton	1 short ton = .907 metric ton
Capacity		
1 liter	= 1.056 U.S.liquid quart	1 liquid quart (U.S.) = 0.946 liter
1 liter	= 0.908 U.S.dry quart	1 dry quart (U.S.) = 1.101 liters
1 liter	= 0.264 U.S.gallon	1 gallon (U.S.) = 3.785 liters
Area		
1 sq. centimeter	= 0.155 sq. in.	1 sq. in. = 6.452 sq. centimeters
1 sq. meter	= 10.764 sq. ft.	1 sq. ft. = 0.093 sq. meter
(or)	1.196 sq. yds.	1 sq. yd. = 0.836 sq. meter
1 are(dkm)^2	= 119.6 sq. yds.	1 sq. mi. = 2.59 sq. kilometers
Volume		
1 cubic meter	= 1.308 cu. yd.	1 cu. in. = 16.387 cu. centimeters
1 cubic centimeter	= 0.061 cu. in.	1 cu. ft. = 0.028 cu. meter
1 cubic decimeter	= 61 cu. in.	1 cu. yd. = 0.765 cu. meter

Bureaus of Standard Weights and Measures

Individual pupils or a small committee should be encouraged to write to each bureau given below to obtain booklets and related literature to be studied by their classmates.

1. The United States Bureau of Standard Weights and Measures Washington, D. C.
2. The International Bureau of Weights and Measures Paris, France

Accuracy and Precision of Measuring

Measurements are not absolutely accurate. The degree of accuracy of a measure and measurement depends upon: (a) the purpose of the measurement, (b) the reliability of the measuring instrument (c) the competency of the measurer, and (d) the size of the magnitude to be measured.

Precision of a measurement depends upon the size of the unit of measure. The smaller the unit of measure used to determine a measurement of a given object or quantity, the greater is the precision of the obtained measurement.

The error of a measurement or the maximum error of an obtained measurement is called the positive error. It is determined by the unit of a measure that is applied. If a foot unit is used to measure a magnitude, then the possible error is one-half a foot. The positive error of measurement is equal to one-half the unit of measure. Therefore, the smaller unit of a measure to be applied will obtain the smallest positive error or the largest amount of precision of a measurement.

The relative error of a measure is obtained by finding the relation or ratio between the measurement and the unit of measure that is applied. The relative error of a measurement is found by dividing the positive error of a measure by the measure itself. The smaller the precision unit in measuring produces the smaller relative unit. The formula used to determine the relative error of a measurement is:

$$Er = \frac{Ep}{m.}$$

(Er is the relative error, Ep is the positive error and m is the obtained measurement.)

WAYS TO OBTAIN A MEASURE AND THE MEASUREMENT

Pupils should be given situations that will enable them to discover and to understand the three ways to obtain the measure and measurement for a given object or quantity:

(a) by counting the number of discrete objects
(b) by direct measurement
(c) by indirect measurement.

The method of counting is used to obtain such measures as the following:

(a) the number of pupils in a class or the number of children in an elementary school
(b) determining the score of a game, such as baseball, volley ball, or a checker game
(c) finding the population of a town or city
(d) finding the number of books in a school library.

Direct measurement is obtained by direct comparison of an object or a quantity with a selected standard unit or measure. The measure of the height of a pupil may be obtained by applying a ruler. The measure of the surface area of a rectangular book may be obtained by selecting a square unit, such as a square inch, and then finding the number of square units needed to cover the interior region of the surface of the book. The measure or amount of liquid in a given container can be found by selecting a unit such as a cup or a pint to dip the liquid from the container to determine the number of units of liquid that were contained in the given container. Pupils should be encouraged to investigate the use of measures in the school, in the home, and in the community, and report to the class the different situations where a measure is obtained by direct measurement.

Pupils should be given situations where a measure cannot be obtained by applying a selected standard unit of measure to a given object or quantity so they may understand the meaning and method of indirect measurement.

If they want to measure the temperature of the room at any given time they must read the relation of the mercury in the thermometer to a scale given on the outside of the thermometer.

When finding the weight of a given object an instrument such as a weighing scale is used to find the relation between an object and the number of standard units required to obtain a balance between the standard units and the amount of the pull of the earth's gravity on the given object. When the scale is in a balanced position the weight measure is obtained by recording the weight of each unit required to produce a balance of the weighing scale.

From their experience in obtaining indirect measurements, pupils should discover the principle that any measurement that is obtained by a calculation is an indirect measurement. Pupils may find the measure of the area of a rectangular sheet of paper by applying the concept of a square inch to each inch of the length of the sheet and then multiplying the result by the number of inches that give the measure of the height of the sheet. If the sheet of paper has a length of 6 inches and a height of 10 inches the area measurement is found by using the formula $A = HL$. (H equals the height in inches and L represents the number of square inches applied to the length.) A proper interpretation of the formula when used to find the area measure is Area $= 10 \times 6$ sq. in. Pupils should explore situations in the classroom, in the home, and in the community to discover the situations requiring the method of indirect measurement and then explain the reason why direct measurement cannot be obtained.

Pupils from their experiments with direct and indirect measurement should develop the following concepts:

1. Linear measure is obtained by applying a one-dimension standard unit to a given line segment between two known points to find the number that is assigned to the line segment.
2. Linear measure of a simple closed plane applies to the measure of the line segments that form the rectangular surface. The formula for finding the measure of the perimeter of a rectangular closed surface is $P = L + L + W + W$. If the object is a square surface then the formula to be used is $P = 4\,S$.
3. Area measure may be found by direct measurement or by indirect measurement.
4. Area measure is obtained by determining the number of standard square units needed to cover the interior surface. (Area is a two dimension measure.)
5. The formula to be developed by the pupils to be used to find area measure is $A = HW$ or $A = LW$.
   (Pupils should understand that the number of square units contained in a row is multiplied by the number of rows.)

MEASURE OF VOLUME

The measure of volume requires the application of a three dimension or cubic standard unit of measure. In the initial stage of experimenting to find volume or cubic measure of a rectangular, three

dimension, container, pupils should apply the direct measure method. They select a standard cubic unit and then determine the number of such units needed to fill the container. If they place cubic units along the length of the base they will determine the number of units required for the first row inside the container. When they apply cubic units to cover the base, the width will determine the number of cubic units in each row on the base, or bottom. By applying layers of cubic units to fill the container they will discover the relation measure between the height and width of the container. By counting the number of cubic units in each layer they then discover the cubic measure of the three dimension container.

From their experiments with direct measure pupils should derive the formula that requires the application of indirect measure and measurement. The formula is $V = HWL$ The measure of the length gives the number of cubic units contained in one row on the base. The measure of width gives the number of rows contained on the base. The measure of height gives the number of layers of cubic units contained in the rectangular object. The formula $V = HWL$ may first be expressed as $V =$ number of layers x the number of rows x the number of cubic units in one row along the length. $V =$ number of layers x the number of rows in each layer x the number of cubic units in one of the rows.

Pupils should develop from their application of direct measure to determine volume or cubic measure an understanding of the formula, $V = HWL$, for computing the volume or cubic measure of any three dimension rectangular container. The formula is based on indirect measure.

It is recommended that cubic measure for pyramids, cylinders, and cones be postponed until the pupils have completed a unit on non-metric geometry and a unit on metric geometry. A guide on non-metric and metric geometry may be obtained from The Piedmont Association for School Studies and Services.

MEASURE OF WEIGHT

The measure of the mass of an object is dependent upon the measure of its gravity. Gravity is the relation between mass and the earth's attraction toward the center of the earth. The measure of the mass of an object is not subject to change; however, the measure of its gravity or weight may change. The weight of an object becomes smaller as the object is removed upward from the surface of the earth.

The measure of the mass of an object is generally obtained by using a weighing instrument called scales. The scales may be balanced scales or spring scales. The relation of the mass of an object on one side of the scales to the standard units used to obtain a balance indicates the weight by reading the count or weight of the standard units applied. Pupils should lift objects and guess their weight before finding the weight measure, in order to obtain a more complete understanding of weight or the earth's gravity attraction.

MEASURE OF CAPACITY

In the United States of America the basic standard unit for liquid measure is the gallon. The cubic measurement of the liquid gallon, as defined by the Congress of the U. S. A., is 231 cubic inches. It is illegal to use the gallon to obtain dry measure. The commonly used units of liquid measure derived from the standard gallon are shown by the following table:

1 Gallon (gal.)	=	4 Quarts
1/2 Gallon (1/2 gal.)	=	2 Quarts
1 Quart (qt.)	=	2 Pints
1 Pint (pt.)	=	2 Cups

There is a legal difference between the U. S. A. gallon and the British gallon. The U. S. A. gallon contains 231 cubic inches while the British gallon contains 277.27 cubic inches.

MEASURE OF DRY MEASURE

In the United States the basic standard unit for dry measure is the bushel. Its measurement is 2150.42 cubic inches. It is not legal in the U. S. A. to apply the standard for dry measure to obtain a liquid measure. The standard unit and sub-units of legalized units for dry measure in the U. S. A. are shown by the following table:

1 Bushel (bu.)	=	4 Pecks
1 Peck (pk.)	=	8 Quarts (dry measure)
1 Quart (qt.)	=	2 Pints (dry measure)

Pupils should be encouraged to pursue research to determine the reason why a dry unit of measure in the U. S. A. contains a larger number of cubic inches than a corresponding unit of liquid measure.

Teachers should encourage pupils to design and to pursue research to find reasons why the legislature assembly of each of the several states in the U. S. A. has enacted legislation to establish bushel weight for such commodities as: apples, beans, tomatoes, and potatoes.

MEASURE OF TEMPERATURE

Temperature is determined by an indirect measure. The measure of a degree of temperature is based on a change in the volume of mercury in a narrow glass tube and an established scale. As heat increases there is an expansion of the mercury in the tube and as heat decreases there is a corresponding decline of mercury in the scaled tube.

There are two types of thermometers used to measure temperature. The most commonly used one is called the Fahrenheit Thermometer. The change of water from liquid to ice, under certain conditions is given the measurement of 32 degrees. The change of water from liquid to boiling is given the measurement of 212 degrees. The scale between these two conditions represents 180 degrees.

Another thermometer that is used in scientific research is called a Centigrade Thermometer. The scale ranges from zero degree (changing point of water to ice) and 100 degrees (changing point of water to gas). The scale between freezing point and boiling point contains 100 degrees.

From their observation and application of the two thermometers pupils should discover and should understand the following properties:

1. One degree on a Fahrenheit thermometer equals
   5/9 degrees on a Centigrade thermometer.
2. One degree on a Centigrade thermometer equals
   9/5 degrees on a Fahrenheit thermometer.
3. A temperature reading on a Fahrenheit thermometer may be converted to a degree reading on a Centigrade thermometer by applying the formula —
   $(F^\circ - 32^\circ\ F)\ 5/9 = C^\circ$.
   $(212^\circ\ F - 32^\circ\ F)\ 5/9 = C^\circ$ **OR** $180^\circ\ F \times 5/9 = 100^\circ\ C$.
4. A temperature reading on a Centigrade thermometer may be converted to a corresponding degree reading on a Fahrenheit thermometer by applying the formula: $C^\circ = 9/5\ C^\circ + 32^\circ\ F$.
   $9/5\ (100^\circ\ C) + 32^\circ = 180^\circ + 32^\circ\ 212^\circ$ Fahrenheit.

It is recommended that eacn classroom be equipped with a Fahrenheit thermometer and a Centigrade thermometer so pupils may take frequent readings of temperature as measured by each instrument and derive an understanding of the scaled relation between the two thermometers. Pupils should be encouraged to pursue research to determine reasons why the Centigrade thermometer is used in scientific research.

MEASURE OF TIME

The system of international units used to measure time was structured by astronomers from their observation and study of the movement of the earth in relation to the sun and the relation of the moon to the earth. Time serves an essential role in the modern world and precision of the measurement of time can easily be observed by observing the measure of time as applied to radio and television programs, and to land and air transportation.

There are **three** important interpretations of time that are to be observed and to be understood by pupils. One interpretation of **time is movement**. Another concept of **time is a period of duration;** the measure from one moment or point of time to another designated moment or point. It is important that pupils also understand that **time moves only in one direction — forward.**

The most frequently used standard units for measuring time which are adopted internationally are given in the following table:

60 Seconds	(sec.)	=	1 Minute
60 Minutes	(min.)	=	1 Hour
24 Hours	(hr.)	=	1 Day
7 Days	(da.)	=	1 Week
4 Weeks	(wk.)	=	1 Month
12 Months	(mo.)	=	1 Year
10 Years	(yr.)	=	1 Decade

It is important that pupils develop an understanding of the meaning and function of standard time zones in the United States and around the world. Pupils from their study of international transportation, trade, and satellite communication are expected to acquire a meaningful understanding of time zones and their function in the measure of time within a given standard zone and the relation of the measure of time around the world. By international agreement there are 24 standard time zones. Each zone is numbered positively and negatively in ordered sequence beginning with the prime meridian that passes through Greenwich, England, and terminates at the 180 degree meridian that is called the International

Date Line. If the time in the Greenwich Zone is known, the time in all the zones in eastern longitude will be later and the time in all the zones in western longitude will be earlier. By international agreement the 180th meridian (the International Date Line) marks the beginning of each day for the whole world. Also by international agreement when a person passes over the 180th meridian into west longitude the time is one day earlier and when a person is traveling west and crosses the International Date Line into east longitude the time will be one day later.

There are six standard time zones in the United States of America:

Eastern Standard Time	(EST)	Central Meridian — 75° West
Central Standard Time	(CST)	Central Meridian — 90° West
Mountain Standard Time	(MST)	Central Meridian — 105° West
Pacific Standard Time	(PST)	Central Meridian — 120° West
Yukon Standard Time	(YST)	Central Meridian — 135° West
Alaska Standard Time	(AST)	Central Meridian — 150° West

When it is one o'clock p.m. in New York City, the corresponding time in Honolulu is five hours earlier, or 8:00 o'clock a.m. When they visit a large airport, pupils should be motivated to observe the six clocks on one of the walls that show the relation of time in the Standard time zones of the United States of America and Standard time zones in other parts of the world.

THE 24-HOUR CLOCK

The most commonly used clock is called a 12-hour clock. The 24-hour day is divided into two subsets of 12 hours each with the beginning of each day in each standard time zone the first subset of 12 hours designed as a.m. (ante meridian) and the second subset of 12 hours designated as p.m. (post meridian). From midnight to noon the morning hours are designated by the symbol a.m. and from noon to midnight the hours are called p.m. hours. Four hours after midnight is expressed as 4:00 o'clock a.m. Four hours after noon is expressed as 4:00 o'clock p.m.

Persons in international transportation, in communications, and in military service use a clock that does not divide the 24-hour period into two periods of 12 hours each. The established 24-hour clock names the hours starting at midnight: Zero to Twenty-four. The 24-hour clock expresses time by the use of four digits. The first two digits on the left indicate the hours after midnight and the last two digits on the right indicate additional minutes. (0500 names 5:00 o'clock after midnight. 1700 names 17 hours after midnight. 0525 names five hours and twenty-five minutes after midnight. 1725 names seventeen hours and twenty-five minutes after midnight.) Many international airlines use the 24-hour clock to name arrival and departure times. It is recommended that pupils be asked: (a) to write for time-tables based on the 24-hour clock and (b) to undertake research to determine the advantage of the 24-hour clock as compared with the 12-hour clock. The expenditure of money to place a 24-hour clock in each classroom can be justified by assuming that by 1980 the 24-hour clock may be more widely used than the 12-hour clock.

ENRICHMENT ACTIVITIES

Pupils should be encouraged to plan and to conduct studies to explore the application of measures and measurement in the fields of business, industry, and the professions. Individual pupils or a team of pupils may experience valuable outcomes from planned visits to such places as: (1) a clothing store (2) a grocery store (3) a hospital (4) an electrical appliance store (5) a radio-television station (6) a building and supply firm (7) a commercial bank. After each exploratory trip, the pupils should be motivated to present a report of their findings to pupils in their classroom.

An improvement in school and community relations may result from the practice of inviting persons from business, industry, and the professions to come to a school to discuss with pupils their dependence upon measures and measuring instruments.

Some of the rapid achievers in a classroom may experience expanded interest when asked to engage in research to find some of the new units of measure that have been invented for application in research, space exploration, jet transportation, and technology and then give a report of their findings to their classmates. The students may prepare Bulletin Board exhibits of their findings.

COMPUTATION WITH MEASURES

It is important that pupils discover and understand that computation is on numbers resulting from measurement. They are expected also to discover and to understand that when writing an equation, vertical algorism, or sentence algorism, involving numbers derived from measurements they use only the numbers and not the units of measure.

For example: $\begin{array}{r} 6 \text{ ft.} \\ + \underline{3 \text{ ft.}} \\ \square \end{array}$ **OR** 6 ft. + 3 ft. = N the computation is: 6 + 3 = 9

ADDITION (Measure)

Addition as applied to measures is similar to addition as an operation on whole numbers or rational numbers. Pupils should be expected to understand that addition of measures is a binary operation on a pair of numbers. The units of measure are not included in the operation of addition.

All the mathematical properties for addition of whole numbers and rational numbers apply also to addition of measures. To obtain the sum for 2 hours 30 minutes and 4 hours 40 minutes, pupils may use the vertical or column algorism or the open sentence or equation form:

Vertical Algorism	**Open Sentence Algorism**
2 hr. 30 min.	2 hr. 30 min. + 4 hrs. 40 min. =
+ 4 hr. 40 min.	2 hr. + 4 hr. + 30 min. + 40 min. =
7 hr. 10 min.	6 hr. + 70 min. =
	6 hr. + (60 min. + 10 min.) =
	(6 hr. + 1 hr.) + 10 min.) =
	7 hr. + 10 min.

SUBTRACTION (Measure)

Pupils are expected to discover and to understand that subtraction as applied to measures is similar to subtraction as an operation on an ordered pair of whole or rational numbers. They must understand that the operation is on an ordered pair of numbers; not on the units of measure. They should also understand that subtraction of a pair of measures is the inverse of addition. 7 lb.—4 lb.=3 lb. 4 lb.+3lb.= 7 lb.) Pupils are expected to apply the additive inverse method when subtracting a given pair of measures. (7 lb. — 4 lb. = $\square$) is rewritten in this form: 4 lb. + $\square$ = 7 lb.

MULTIPLICATION (Measures)

It is recommended that multiplication as applied to measures be limited to examples where the first of an ordered pair (the multiplier) is a whole number, a rational number, or a mixed number, and the second of the ordered pair is an obtained measure. (4 x 6 ft. = $\square$). This type of multiplication is really a process of rapid addition of a given number of measures.

Since multiplication is an operation on ordered pairs of numbers, pupils are expected to discover that the mathematical properties for multiplication of whole or rational numbers apply also to multiplication on measures.

The problem expressed by the open sentence 3 ft. x 4 ft. = $\square$ relates to finding area. If problems of this type are to be included in the curriculum, then it is recommended that pupils re-write the problem in the form 3 x 4 sq. ft. = $\square$

DIVISION (Measures)

Pupils should be expected to discover and to understand that the mathematical concepts that they have learned for division of whole numbers and rational numbers can be applied to division of measures. It is important that they understand that the operation of division applies to the numbers and not to the units of measure.

There are two types of questions that can be answered by division:

Question A: 18 ft. ÷ 3 = $\square$ The question to be answered is:
"If 18 ft. is arranged in 3 equal size subsets,
then how many measures will be in each subset?
18 ft. ÷ 3 ft. = 6 ft. 18 ÷ 3 = 6 There will be 3 equal sets of 6 ft. each.

Question B: 18 ft. ÷ 3 ft. = $\square$ The question asked is:
"If 18 ft. is rearranged into subsets of 3 ft.
each, then there will be how many subsets of 3 ft.?"
18 ft. ÷ 3 ft. = $\square$ Is to be interpreted as
18 ft. ÷ 3 ft. = 6 for reason
that division applies only to numbers.
There will be 6 subsets of 3 ft. each.

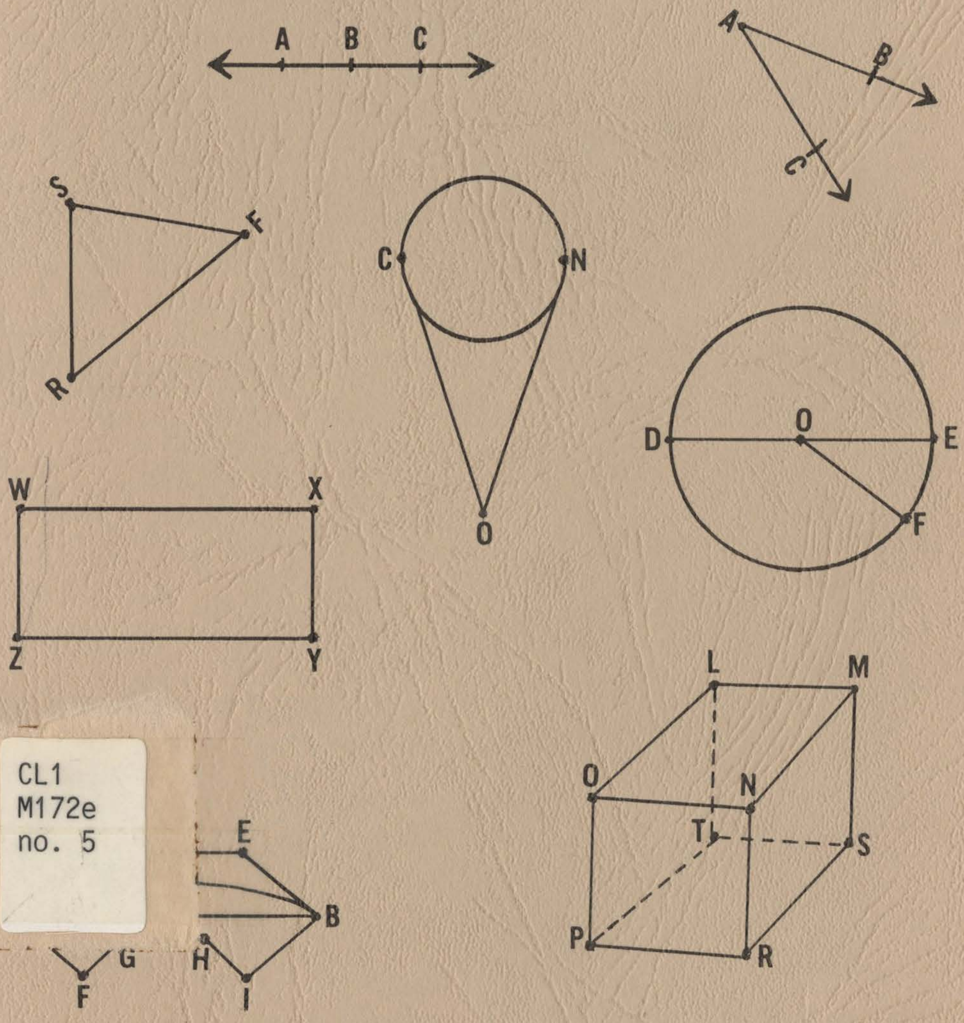
THE PIEDMONT ASSOCIATION
FOR
SCHOOL STUDIES AND SERVICES

GUIDE NUMBER FIVE

elementary mathematics

informal geometry · non-metric and metric
mathematical concepts · terms · properties

E. T. McSwain



ELEMENTARY MATHEMATICS

Guide One — Sets • Whole Numbers

Guide Two — Selected Numeration Systems

Guide Three — Rational Numbers

Guide Four — Measures and Measurement

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1967

INTRODUCTION

Four of the basic units in the mathematics curriculum for the elementary school and for grades seven and eight in the junior high school are: 1) SETS 2) NUMBERS 3) INFORMAL ALGEBRA, and 4) INFORMAL GEOMETRY. From their study of sets, pupils should have developed a cumulative understanding of the mathematical concepts of numbers and the mathematical properties related to operations on pairs of numbers. They should have discovered and understood reasons why the mathematical properties for addition and multiplication on natural numbers apply also for addition and multiplication on pairs of rational numbers and pairs of integers.

During their study of numbers and operations on pairs of numbers, pupils should be expected to discover the meaning and function of informal algebra. From their work with closed and open mathematical sentences, pupils begin to interpret and understand that informal algebra is a kind of generalized arithmetic which employs all the symbols of arithmetic and the letters of the English alphabet. The letters constitute the set of literal numbers. Pupils are expected to discover and understand the relation to and difference between the mathematical language of arithmetic and of informal algebra. In the open sentence, $8 + 14 = \square$, there is only one standard number for $8 + 14$.

Pupils, when reading, $a + b = 22$, should interpret "a" to be a variable and "b" to be a variable. They are expected to understand the mathematical reasons why there are many pairs of numbers that have 22 as the sum. Experiences with open mathematical sentences or conditional equations enable pupils to discover and understand the application of the language and operations of informal algebra.

Informal geometry gives pupils the opportunity to interpret geometry as a branch of mathematics involving the application of geometric concepts to space and to the shape and size of physical objects or models which occupy space. Recent experimental investigations reveal findings that girls and boys in the elementary school can experience interest in and acquire basic concepts involving informal geometry. The intuitive method in observing and examining physical objects or drawings is an effective way for pupils to discover, interpret, and comprehend many geometric concepts called:

Point	A Straight Line
Set of Points	A Line Segment
Space	A Geometric Curve
Plane	A Plane Geometric Solid

Non-Metric geometry refers to geometric concepts that do not involve measurement.

Metric geometry refers to the instruments and operations used in determining the measurement of the size of physical objects or geometric figures or models.

Pupils are expected to apply precision when expressing geometric concepts and describing the geometric characteristics of physical objects or models as they begin and continue exploration of non-metric geometry.

For Example: • A geometric point is the mental concept or abstraction of a fixed position or location in space. It cannot be seen or moved. The symbol generally used to name or express a geometric point is a "dot" or •; or a capital letter and a dot, "$\underset{\bullet}{A}$".

A geometric point is very, very small and it is not assigned a measure. Space is interpreted as containing an infinite set of points.

- A geometric line segment is a designated subset of points contained in a set of points that constitute a straight line and having two fixed or named points; a beginning point and an end point. A geometric line segment is a mental abstraction; it exists only in a person's mind. The symbols, $\overline{AB}$ or $\overset{A}{\bullet}\!\!-\!\!-\!\!-\!\!\overset{B}{\bullet}$ are used to name the concept of a specific geometric line segment.

Pupils should be motivated to observe, to explore, and to think about physical objects or figures found in the classroom, in the home, and in the community; to discover and to describe the geometric concepts which may be applied to each of the physical objects or drawings. The learning experiences should be dependent upon intuition, exploration, experimentation, and precision in thinking about or describing geometric concepts related to the shape or size of physical objects or models which occupy space.

PUPIL EXPLORATORY-DISCOVERY versus TEACHER-DIRECTED METHOD

Psychological investigations and instructional experimentation support the generalization that the most creative, interesting, and productive process in discovering, interpreting, and understanding geometric concepts and properties related to geometric figures or models, involves observing, examining, questioning, reasoning, and constructing derived generalizations. Less meaningful are conceptual learnings based upon teacher-directed explanations and heavy dependence upon a textbook. When given the opportunity to observe and examine physical objects or geometric models, pupils may experience self-directing motivation in determining appropriate geometric responses to such questions as:

- What is the geometric shape of the physical object or model
- What set of points are involved
- What intersection or union of line segments forms the geometric shape of the physical object or model?
- Is the shape of the object or model a geometric plane or a geometric solid?
- Are angles contained in the geometric concept of the shape of the physical object or figure?
- What geometric postulates and terms are needed to describe the shape of the object, model, or drawing?
- How may the measure of the size of the physical object or model be determined?

Pupils may discuss their responses to such questions given above to appraise the quality and precision of the derived findings. If additional assistance is desired pupils may seek assistance from their teacher or refer to information to be obtained from reading a textbook or resource reference. Cumulative progress in interpreting and understanding geometric concepts and related terms and postulates is more meaningful when teachers and pupils recognize the productive value of intuitive exploration, cognitive reasoning, and meaningful structuring of derived generalizations.

INTUITIVE EXAMINATION OF GEOMETRIC MODELS

Properly constructed models of geometric concepts and shapes should be readily available in each classroom. Geometric models are more appropriate for examination when they are constructed by using strips of rigid wire or strips of wood. A geometric model so constructed enables pupils to observe and interpret more readily the geometric concepts involved.

For Example: Pupils when examining a geometric triangle constructed of strips of wire or wood may discover more readily the three line segments; the three union points of the line segments; the set of points which form the interior region; the set of points which constitute the exterior region of the triangle; and the interior angles of the geometric model.

Geometric models may be constructed by using pieces of plastic material, joined by using library tape. Such geometric models have many advantages over pictures of physical objects or geometric models. Capable students in the junior high school may find interest and satisfaction in preparing a set of geometric models to be given to pupils in a primary grade. Each model should have a label that names the term given to the model. If desired, a label may be attached to each geometric model giving the name of the constructor and donor. Similarly a set of well-constructed drawings or figures of geometric concepts and terms may be prepared and presented for use in a primary classroom. Rapid achievers in an intermediate grade may derive interest and satisfaction from constructing a set of geometric models or a set of geometric drawings to be given to a group of pupils or a classroom.

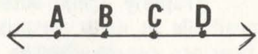
GEOMETRIC CONCEPTS — POINTS • LINES • CURVES • RAYS

1. **Geometric Point.** A geometric point is a mental concept or abstraction of a given and fixed position or location in space. It cannot be seen or moved. The concept of a point exists only in a person's mind. A geometric point is an undefined term. It is so small that it is not assigned a measurement. The symbol generally used to name or designate a given point is a "dot" made by a pencil, piece of chalk or a fountain pen. Another symbol that may be used to name a point is "$\underset{\bullet}{A}$".

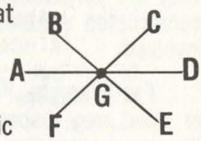
2. **A Geometric (Straight) Line.** The concept of a geometric straight line is related to an infinite set of points arranged in a straight line or pattern. It does not possess a beginning point or an end point. Two fixed points in the set of points are required to form a geometric straight line. The symbol generally used to name a given geometric straight line is $\overleftrightarrow{AB}$. The letters "A" and "B" identify two given points of the set of points and the symbol "$\longleftrightarrow$" refers to a line that is between A and B and continues to the right and to the left of points A and B.
   - Two designated points in a set of points are needed to derive the concept or abstraction of a given geometric (straight) line in space.
   - Two distinct points are needed to describe only one straight line.

3. **A Geometric Line Segment.** A geometric line segment refers to the concept of a subset of points contained in a set of points that form a straight line. Two of the points in the subset of points are used to designate the beginning point and the ending point of a line segment. The symbol $\overline{AB}$ is used to name a given line segment. Letter A names the first or beginning point and the letter B names the end or terminating point. The bar above the letters A and B refers to the line segment formed by the subset of points starting with point A and ending with point B.

- A geometric line may contain many line segments provided each line segment is identified by a given subset of points.



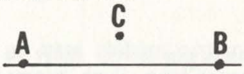
- Many geometric line segments may contain a given point that is described as the intersection point.



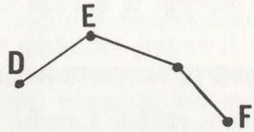
- Points contained in a set of points that form a geometric line or curve are called **collinear** points. Any point that is not contained in a given set of points that form a geometric line or curve is called a **noncollinear** point:

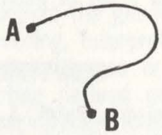
  Points A and B are **collinear.**

  Point C is **noncollinear.**

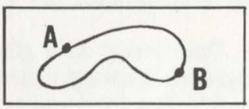


4. **A Geometric Curve.** A simple geometric curve is interpreted as a set of points in a plane containing a given starting point and a given end point. A geometric curve may be illustrated by the path made by a pencil or pen as it moves from a starting point and stops at a designated end point. The geometric curve is called a simple open curve when the end point of the curve is not the same point as the starting point.



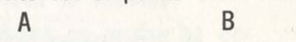


A simple closed curve is formed by a subset of points in a plane and the beginning point and the end point are identical and there is no intersection of the closed curve. The figure shown here illustrates the concept of a simple closed curve in a plane.



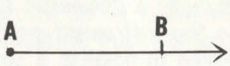
From their examination of geometric figures which illustrate the concept of a simple closed curve, pupils should discover that the simple closed curve separates the plane into two subsets of points. One subset of points represent the interior region of the simple closed curve and the other subset of points represent the exterior region of the simple closed curve contained in a given plane.

5. **A Geometric Ray.** A geometric ray is interpreted as an infinite set of points that form a straight line but the set does not contain an end point.

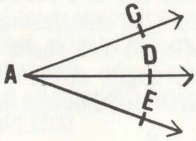


In this illustration A denotes the beginning point or point of origin of the geometric ray and the arrow indicates the concept that the ray continues. Pupils may improve their understanding of a geometric ray if they think of a beam of light starting with a lighted candle, or starting from a headlight of an automobile. When they observe that many beams of light start at the lighted candle or at the headlight of an automobile they may quickly comprehend the concept that an infinite set of geometric rays may begin at a given point.

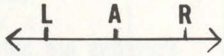
- Symbol $\overrightarrow{AB}$ illustrates the concept of a ray beginning at Point A.



- Symbols $\overrightarrow{AC}$ $\overrightarrow{AD}$ $\overrightarrow{AE}$ illustrate the concept of three rays that begin at point A.



- Symbol $\overleftrightarrow{LAR}$ illustrates the conçept of two rays beginning at point A, but moving in an opposite direction. A geometric ray may be interpreted as a subset of a given geometric line.



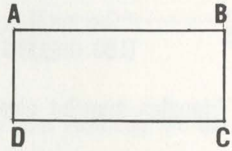
6. **Measure of Geometric Line or Ray.** A geometric line segment or a segment of a geometric ray has only one dimension, **length.** Pupils should use a ruler containing a given number of standard units on the edge, arranged in an ordered sequence, to determine the measurement of a geometric figure representing a given line segment or a given segment of a ray. They are expected to understand the difference in meaning of a unit of measure and the symbol used to name the measurement of a given line segment or segment of a ray. The measurement, 11 inches, is to be interpreted:

   - The numeral 11 names the count of unit of measure used.
   - The word inches names the kind of standard unit of measure that was applied, one inch.

GEOMETRIC PLANES AND ANGLES

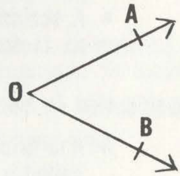
A geometric plane is described as an infinite set of points contained in a flat surface portion in space. A subset of a geometric plane is described or thought of as a plane figure. Pupils may be motivated to answer the questions:

- Why do three noncollinear points form a geometric plane figure?
- Why do four or more noncollinear points form a geometric plane when the pairs of line segments joint at one of the noncollinear points?



From their observing and intuitive thinking, pupils are expected to interpret and understand the structure of a geometric plane figure that is called an angle. A geometric angle is formed by two geometric rays which begin at the same point.

The angle AOB is formed by the two rays $\overrightarrow{OA}$ and $\overrightarrow{OB}$ which begin at the point O, or the common point. The common point of origin is called the vertex of the angle. Each of two rays which forms an angle designates a side of a given angle. In this illustration the ray OB is one side of the angle AOB and the ray OA is the other side of the angle.

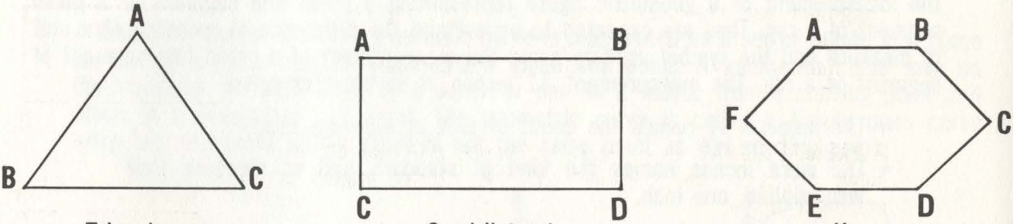


Any angle in a given plane separates the set of points into two subsets. One subset of points form the interior of the angle and the other subset of points form the exterior of the angle. When the sides of an angle are the same ray then the angle is interpreted as a zero angle. A straight edge or ruler and a compass are the instruments used in constructing or

drawing a geometric figure that represents the concept of a geometric angle. A protractor is the instrument used to determine the measurement of an angle. Each pupil should be asked to possess a straight edge or rule, a compass, and a protractor. From their observation and use of a protractor, pupils will discover and comprehend the reason why the unit of measure, in determining the measurement of an angle, is a degree. When measuring angles of different size pupils may soon discover the meaning of an **acute** angle, a **right** angle, an **obtuse** angle, and a **straight** angle. When examining physical objects pupils may quickly discover the different kinds of angles that help to form the shape of an object.

POLYGONS

Polygons are simple closed curves that contain only line segments. Each segment is called a side of the polygon. The point that represents the union of two adjacent sides is called a **vertex** of the polygon. A polygon has as many vertices and angles as the number of sides or segments.



Triangle   Quadrilateral   Hexagon

1. **Triangles.** A triangle is a three-sided polygon. It contains three sides (each a line segment) three angles, and three vertices. The sides of a triangle need not be congruent (of the same length) and the interior angles need not be of the same size. By applying a protractor to the figure of a triangle, pupils may readily understand the total measurement of the three angles is 180 degrees. Pupils may experience interest in discovering the meaning of the formula:

   "The total measurement of the angles of a polygon equals N-2 (180 degrees.)"

Triangles may be classified in terms of their angles or their sides.

Classification in terms of sides:

- An **equilateral triangle** contains three congruent sides.
- A **scalene triangle** contains no congruent sides.
- An **isosceles triangle** has two sides which are congruent.

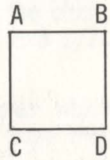
Classification in terms of angles:

- If a triangle has one angle with a measurement of 90 degrees, it is called a **right-angle triangle.**
- If a triangle has one angle with a measurement greater than 90 degrees it is called an **obtuse-angle triangle.**
- When each of the angles of a triangle is less than 90 degrees, the triangle is called an **acute-angle triangle.**

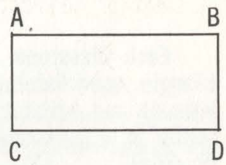
Pupils should be encouraged to use a ruler, a compass, and a protractor to construct or draw geometric figures to illustrate triangles included in each classification. Pupils should discover the meaning of the formula Area $\triangle = \frac{1}{2}$ HB.

2. **Quadrilateral Polygons.** A quadrilateral polygon or a quadrilateral is a geometric figure that is formed by four line segments and the union of the adjacent sides. Each quadrilateral contains four vertices and four interior angles. Quadrilaterals are classified according to the relationship of the opposite side. The most frequently observed figures are: SQUARE • RECTANGLE • PARALLELOGRAM • TRAPEZOID

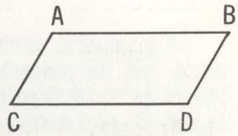
A **SQUARE** is a geometric figure containing four congruent sides. The opposite sides are parallel and each interior angle is a right angle.



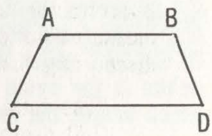
A **RECTANGLE** is a geometric figure with the opposite sides congruent and the interior angles are right angles.
Sides AB and DC are congruent and parallel.
Sides AC and BD are congruent and parallel.



A **PARALLELOGRAM** is a geometric figure with the opposite sides congruent and parallel, but the interior angles are not right angles.
Sides AB and CD are congruent and parallel.
Sides AC and BD are congruent and parallel.
Angles BAC and BDC are congruent obtuse angles and angles ACD and ABD are congruent acute angles.



A **TRAPEZOID** is a geometric figure with two opposite sides parallel but not congruent, and the other two sides are not congruent and are not parallel.
Sides AB and DC are parallel but not congruent.
Sides AC and BD are not parallel and are not congruent.



Pupils may experience interest in using a protractor to discover the meaning of the formula:

- The sum of the measurement of the interior angles of a quadrilateral equals N—2(180°). 4—2(180°) = 2(180°) = 360 degrees.

The measurement of the interior region or surface of a quadrilateral is determined by selecting and applying a square unit of measure. Pupils should be encouraged to explore to find the precise answer to the questions:

- Why does the interior region or surface of a quadrilateral have two dimensions; 1) length, and 2) width or height?
- What is the meaning of the formula "The Area of a quadrilateral equals BW or BH?"

Regular polygons may be classified according to their number of sides. The sides of a regular polygon are congruent and the interior angles are congruent. Pupils should examine

regular polygons and by applying a protractor find the meaning of the formula, "The measurement of each angle in a regular polygon equals $\frac{(N\text{-}2)\ 180^\circ}{N}$."

Regular Polygons

Name	**Number of Sides**	**Measurement of each Angle**
Square	4	90 degrees
Pentagon	5	108 degrees
Hexagon	6	120 degrees
Septagon	7	128 4/7 degrees
Octagon	8	135 degrees
- - - - - -	- - -	- -
N gon	N	$\frac{N\text{-}2\ (180^\circ)\ \text{degrees}}{H\ \text{degrees}}$

Each classroom should possess at least one set of models and geometric figures: triangle, quadrilateral, pentagon, hexagon, septagon, and octagon. Rapid achievers may find interest and satisfaction in constructing models and/or drawings of polygons to give to pupils in a classroom where there is not a set of models and/or a set of geometric drawings.

CIRCLE

A geometric circle is a simple closed curve possessing specific properties. A geometric circle may be properly interpreted as a set of points that form a simple closed curve with the points contained in the set being of equal distrance from a center point in the interior region. A geometric circle divides a plane into a subset of points which form the interior region and the subset of points which are in the exterior region. A line segment that begins at one point in the circle and passes through the center point and ends in a point opposite the beginning point is called the **diameter** of the circle. A line segment that begins at the center point and ends at a point in the circle is called a **radius.** Pupils should be encouraged to discover under what conditions the union of two radii equals a diameter. A subset of points in the circle form a curve that is called a geometric **arc.** The perimeter is the term which names the length of the circle. A **perimeter** is sometimes called a **circumference.**

$\overline{ACB}$ is the diameter

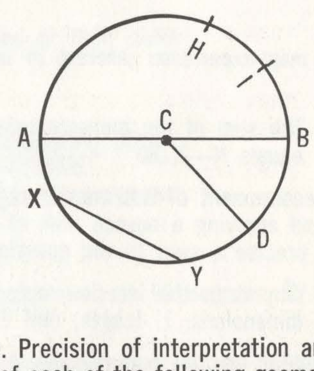
$\overline{CB}$ is a radius

$\overline{CD}$ is a radius

$\overline{HI}$ is a geometric arc

BCD is a central angle

$\overline{XY}$ is a chord



A semicircle is equal to one-half a given circle. Precision of interpretation and description requires pupils to discover the proper meaning of each of the following geometric terms: a Circle, Center point contained in the interior region of a circle, Diameter, Radius, Radii, Arc, Chord, Central Angle of a Circle, Interior region, Exterior region, Semicircle, and Circumference.

The measurement for a circle, a diameter, a radius, a chord, or an arc is found by applying a standard unit of length. The measurement for the interior surface or region of a circle is found by applying a square unit. Pupils may interpret more clearly the meaning of the measurement of a given circle by using a compass and drawing the figure of a circle on a sheet of graph paper that contains a set of inch squares. Pupils should not be expected to show proof of the formula: "Area $\bigcirc = \pi r^2$." Rapid achievers may experience interest in determining the meaning of Pi, or π. By drawing a straight line congruent in length with the length of the circumference of the circle and drawing another line congruent to the length of the diameter and then comparing the straight line, representing the circle, with the line representing the diameter, pupils may discover the meaning of Pi (π) $= 3\ 1/7C$ or $\pi = 3.1416\ C$.

By using a compass, pupils should be motivated to draw geometric figures which illustrate the concept of a circle. Then by applying a protractor and a straight edge they should name a diameter, a radius, a chord, and a central angle.

CONGRUENCE AND SIMILARITY

Pupils who have many experiences in observing, handling, and examining geometric models and figures may begin to develop concepts which will help them to give an appropriate answer to two questions concerning the comparison of two geometric models and figures:

- When are two geometric figures congruent?
- When are two geometric figures similar?

From intuitive study of geometric models or figures, pupils may soon develop the concept that two geometric figures or models having the same size and shape are considered to be congruent. They may discover also that two geometric figures having the same shape but not having the same size are considered to be similar.

Pupils should be given many opportunities to pursue intuitive study of geometric figures and physical objects that have geometric shape and size so they may develop concepts to be used in giving proper answers to each of these questions:

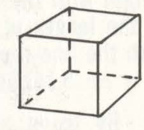
- When are two geometric line segments congruent?
  When are two geometric line segments similar?
- When are two geometric angles congruent?
  When are two geometric angles similar?
- When are two geometric triangles congruent?
  When are two geometric triangles similar?
- When are two geometric polygons congruent?
  When are two geometric polygons similar?

GEOMETRIC SOLID FIGURES

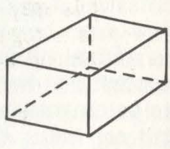
It is suggested that in the elementary school the intuitive, exploratory study of solid figures be limited to cubes, regular rectangular prisms, regular triangular prisms, regular pyramids, right cylinders, right cones, and the sphere. Pupils may be given many situations in which they are asked to examine solid physical objects and geometric solid figures for the purpose of applying previously learned geometric concepts to determine what geometric concepts and properties are related to solid figures. Each classroom should have several

sets of solid physical objects and geometric models and figures of solids. Pupils should be encouraged to work as individuals or in small groups and then report their findings to their peers. The teacher's method of instruction should emphasize motivating questions and activities and limit the time spent in telling.

1. **Geometric Cube.** A cube has six congruent lateral faces or surfaces, each being a geometric square. The intersection of any two faces is a segment called an edge. The intersection of two segments is a point called a vertex. The opposite faces are parallel. By using a protractor pupils will discover that each of the interior angles is a right angle. Careful examination will enable them to discover that a cube divides a set of points in space into three subsets. One subset of points is the **interior** region; another the **exterior** region; and the third is the **solid** figure. A teacher may ask the pupils to construct six square planes or surfaces and then, by applying library tape, to construct a cube. Pupils may be asked to use a ruler and a protractor and make a drawing of a cube.



2. **Rectangular Prism.** When examining a regular rectangular prism and applying previously discovered geometric concepts, pupils may experience little difficulty in learning the following concepts applied to a regular rectangular prism:
   - The prism contains six lateral faces or surfaces. Each face is a rectangle.
   - The two end faces are congruent rectangles and the other four faces are congruent, but may be larger in size than the two end faces.
   - The intersection of two faces is a line segment called an edge.
   - The intersection of two segments is a point called a vertex.
   - A regular rectangular prism has eight vertices and twelve edges.
   - The interior angles are right angles. The opposite faces or surfaces are parallel.



   Pupils may experience motivation and interest from the question:

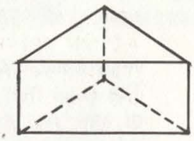
   "Can you construct six rectangles and determine, by using them, if you can construct a regular rectangular prism?"

   Pupils may be asked, "Use a ruler and a protractor and determine if you can draw a figure of a regular rectangular prism."

   Another question that may challenge the pupil's observation and thinking is, "A regular rectangular prism divides a set of points in space into how many subsets of points?" Pupils may be asked to bring to the classroom a few physical objects which have the shape of a regular rectangular prism.

3. **Regular Triangular Prism.** When the polygon that outlines the base of the prism is a triangle, the prism is called a regular triangular prism. When observing and examining models of triangular prisms and drawings of triangular prisms, pupils should acquire an understanding of the following geometric concepts and properties:
   - The polygons which outline the base surface and the top surface are **triangles**.

- The polygons which outline the sides or lateral faces are **quadrilaterals.**
- The intesrection of two adjacent faces or sides is a line segment that is called an **edge.**
- The intersection of two edges is at a point, called the **vertex.**
- The set of points which form a regular triangular prism contains three subsets of points: one determines the **interior region**; one the **exterior region**; and one forms the **frame** of the figure.
- The bases, or bottom and top surfaces, are parallel.
- The three lateral faces are quadrilateral polygons.
- There are six vertices.
- There are nine line segments.
- The union of the interior region and the simple closed surface forms a solid region.



Pupils should be asked to construct a set of triangular polygons of different sizes and a set of quadrilateral polygons of different sizes and then experiment or determine how many regular triangular prisms they can make or form.

Pupils may develop a meaningful understanding of the measurement of the interior region of a regular rectangular prism and the interior region of a regular triangular prism by performing an experiment.

First they prepare a set of rectangular polygons from thin strips of cardboard.

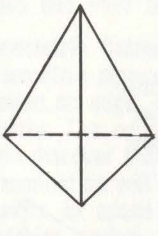
Next they construct a regular rectangular prism by joining the strips, using library tape. Then they construct a set of inch cubes.

Then they place the cubes in the interior region of the prism until the interior space is filled. Then they pour out the cubes and count them. The count will name the cubic measurement.

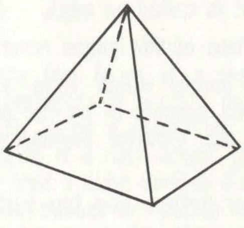
Pupils may soon discover that the length of the prism indicates the number of cubes in one row at the bottom; the width will indicate the number of rows of cubes in the bottom layer; and the height will indicate the number of layers. They then may comprehend the meaning of the formula: $V = H \times W \times L$ or $V = HWL$.

After completing this experiment, pupils may then divide the prism into two congruent triangular prisms. After filling one of the rectangular prisms with inch cubes, they will discover that it contains one-half the number of cubes required to fill the interior region of the regular rectangular prism or solid. Pupils may then acquire an understanding of the statement, "The volume of a regular rectangular prism equals one-half the length of the base by the width by the height. Volume of a regular rectangular solid equals $\frac{1}{2}$ HWL.

4. **Pyramids.** It is suggested that the pupil's intuitive study of regular pyramids be limited, at first, to regular rectangular pyramids and triangular pyramids.



Triangular Pyramid



Rectangular Pyramid

A pyramid is a simple closed surface containing a given set of triangular lateral faces. The base may be a quadrilateral, or a triangle, or any shaped polygon. The intersection of any two faces is a segment called an edge. The intersection of three, four, or more edges at a point, not in the same plane as the base, is called the **apex.** Pupils should be expected to apply geometric concepts learned from their study of regular prisms when discovering the geometric concepts related to regular rectangular pyramids and regular triangular pyramids, such as:

- Base of a pyramid
- The faces of a pyramid
- Edge
- The vertex
- A pyramid is formed by the union of all the faces.
- Pyramids are classified according to the shape of the polygon that forms the base.

Pupils should be encouraged to conduct experiments to find if they can understand that the volume of a pyramid is equal to the volume of a rectangular solid that has the same dimensions as the pyramid.

5. **Right Cylinders.** It is suggested that the intuitive study of cylinders be limited, at first, to right cylinders. A right circular cylinder is formed by the union of a simple closed circular surface or curve and two congruent circular bases. Perpendicular line segments between points on the circle of each base form the simple closed circular surface. The geometric concepts to be related to a right cylinder are:

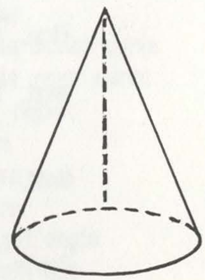
- Each base contains a set of points which form the circle and a set of points which form the interior surface of the base.
- A set of segments are perpendicular and have their end points at a point in the circle of each base.
- A set of points form the interior region of the cylinder.

Pupils can improve their understanding of the volume of a right cylinder by reviewing the method used to find the volume of a rectangular solid. The unit of measure is a cube. Pupils first find the number of cubes needed for the base or bottom layer. The

number of layers is determined by the measurement of the height of the cylinder. The number of cubes in the bottom layer is multiplied by the number of layers determined by the height of the cylinder. The formula used in finding the volume of a cylinder is V = BH. (B is the area of the base, and H is the height.)

6. **Right Cone.** From their intuitive observation and study of models and geometric figures of a right cone, pupils should discover how a cone is related to a cylinder as a regular pyramid is related to a prism. A cone is a region bounded by a simple closed circular surface. The vertex is not in the plane in which the circular base is located. All line segments contained in the simple closed surface have one end point that is the vertex, the other end point is the set of points which form the circle of the base. From their intuitive study of models and figures of right cones, pupils derive these geometric concepts:

- the Vertex
- the Lateral Surface
- the Set of Segments which form the Lateral Surface
- the Circle of the Base
- the set of points which form the Interior Region of the Base
- the set of points which form the Interior Region of the Cone
- One model of a right cone that is familiar to pupils is an ice cream cone.



Pupils may improve their interpretation of the volume of a right cone by undertaking this experiment:

First, obtain three ice cream cones which have the shape of a right cone.

Then, construct a right cylinder that has a height equal to the height of the ice cream cone and a base equal to the size of the base of the ice cream cone.

Next, fill each ice cream cone with sand and then empty them into the right cylinder.

Pupils will discover that three ice cream cones filled with sand, when poured into the cylinder, will fill the cylinder.

From this experiment pupils can develop the concept that the volume of a right cone equals one-third of a right cylinder with dimensions equal to the dimensions of one of the ice cream cones. Pupils may then derive the formula: $V = 1/3\ \pi\ r^2H$. (The volume of a cone is one-third the volume of a right cylinder having the same dimensions.)

GEOMETRIC SPHERE

It is suggested that each pupil obtain a rubber ball. (The set of rubber balls should be of the same dimension.) From their intuitive examination of the rubber ball, pupils should begin to develop concepts related to the surface of the ball that is a model of the surface of a sphere. Next, they use a sharpe knife and divide the ball into two equal parts. From careful examination, pupils may soon discover the concept of a diameter of the ball and its relation to the diameter of a sphere. When they slice half of the ball into sections in a horizontal manner, pupils may begin to discover the meaning of circles of latitude of the earth's sphere. When pupils slice the other half of the ball in a vertical manner they may discover ideas which add meaning to the term, a line of longitude of the earth's sphere.

An experiment is suggested which, if executed by pupils may help them to interpret the meaning of the volume of a sphere.

First, they obtain a ball or a plastic sphere.

Next, they construct a cylinder that has a diameter and height equal to the diameter of the ball or plastic sphere.

Next, they fiill the cylinder with water. Then slowly lower the ball or sphere into the cylinder.

Pupils will observe that some of the water will overflow as the ball or sphere is lowered into the cylinder.

When the ball or sphere is removed from the cylinder, pupils will discover that the cylinder is only one-third filled with water. Pupils should discover that the ball or sphere displaced two-thirds of the volume of the cylinder.

If the volume of the sphere equals π r2H (H $=$ 2r) then the volume of the sphere is $2/3 \times \pi r^2 2r$ **or** $4/3 \pi r^3$.

GLOSSARY

1. Acute angle
2. Adjacent angles
3. Altitude
4. Angle
5. Apex
6. Arc
7. Area of a circle
8. Area of a polygon
9. Base of a geometric figure
10. Bisect
11. Center point
12. Central angle
13. Chord
14. Circle
15. Circumference
16. Collinear
17. Compass
18. Cone
19. Congruence
20. Congruent
21. Cube
22. Cylinder
23. Diagonal
24. Diameter of circle
25. Degree
26. Decagon
27. Edge
28. Equilateral triangle
29. Exterior region
30. Geometric figure
31. Geometry
32. Height
33. Horizontal
34. Informal geometry
35. Interior region
36. Isosceles triangle
37. Lateral face or surface
38. Latitude
39. Length
40. Line segment
41. Longitude
42. Measure
43. Measurement
44. Metric geometry
45. Noncollinear
46. Non-metric geometry
47. Obtuse angle
48. Octagon
49. Parallel lines
50. Parallelogram
51. Pentagon
52. Perimeter
53. Perpendicular lines
54. Pi
55. Plane
56. Point
57. Polygon
58. Protractor
59. Pyramid
60. Quadrilateral
61. Radius, Radii
62. Ray
63. Rectangle
64. Rectangular prism
65. Right angle
66. Right triangle
67. Ruler
68. Scalene triangle
69. Segment
70. Semicircle
71. Set, empty
72. Set of points
73. Side
74. Similar
75. Simple closed curve
76. Simple open curve
77. Solid region
78. Space
79. Sphere
80. Square
81. Straight angle
82. Straight line
83. Subset of points
84. Symbol
85. Trapezoid
86. Triangle
87. Triangular prism
88. Triangular pyramid
89. Vertex, Vertices
90. Volume
91. Volume, of cone
92. Volume, of cylinder
93. Volume, of prism
94. Volume, of pyramid
95. Volume, of sphere

NOTE: Pupils during their cumulative study of informal geometry are expected to discover and to understand the precise meaning of the geometric terms listed in the Glossary. Precision in using the language of geometry is very important.

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